

Pathwise uniqueness for stochastic reaction-diffusion equations in Banach spaces with an Hölder drift component*

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Abstract

We prove pathwise uniqueness for an abstract stochastic reaction-diffusion equation in Banach spaces. The drift contains a bounded Hölder term; in spite of this, due to the space-time white noise it is possible to prove pathwise uniqueness. The proof is based on a detailed analysis of the associated Kolmogorov equation. The model includes examples not covered by the previous works based on Hilbert spaces or concrete SPDEs.

We prove pathwise uniqueness for a general class of reaction-diffusion equations in Banach spaces with an Hölder drift component, of the form

$$\begin{cases} dX(t) = [AX(t) + F(X(t)) + B(X(t))]dt + dw(t) \\ X(0) = x. \end{cases}$$

Here A is the Laplacian operator in the 1-dimensional space domain $[0, 1]$ with Dirichlet or Neumann boundary conditions, the Banach space E is the closure of $D(A)$ in $C([0, 1])$, $x \in E$, F is a very general reaction-diffusion operator in E which covers the usual polynomial nonlinearities with odd degree, having strictly negative leading coefficient, $B : E \rightarrow E$ is only Hölder continuous and bounded, $w(t)$ is a space-time white noise. See the next section for more details, in particular about the assumptions on F .

For finite dimensional stochastic differential equations it is well known that additive non degenerate noise leads to pathwise uniqueness in spite of the poor regularity of the drift (see [19], [16] among others). Due to a number of relevant open problems of uniqueness for PDEs, there is intense research activity to understand when noise improves uniqueness in infinite dimensions (see [10] for a review). Our result, which applies to a large class of systems of interest for applications, contributes to this research direction.

The present paper is the first one dealing with the problem of pathwise uniqueness in Banach spaces instead of Hilbert spaces. This extension introduces many difficulties and does not represent a mere generalization of the previous cases studied in the existing literature. We

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treat here the concrete case of the Banach space $E = C([0, 1])$ or $E = C_0([0, 1])$ (depending on the boundary conditions). A typical tool in Hilbert spaces is the finite dimensional projection or approximation by means of the elements of an orthonormal basis. Here we implement the idea recently developed in [6] of using an orthonormal basis of the Hilbert space $L^2(0, 1)$ made of elements which belong to E . This method allows to perform certain finite dimensional approximations and in particular to write Itô formulae for certain quantities; the control of many terms is often nontrivial but successful.

This paper is, in a sense, the generalization of [8] to Banach spaces (see also [9] on bounded measurable drift and the work in finite dimensions [11] where part of the technique was developed in order to construct stochastic flows). From the viewpoint of examples, this generalization is relevant. Both the reaction-diffusion term F and the Hölder term B are not covered by [8] except for particular cases. One could naïvely think that it is sufficient to apply a cut-off and reduce (locally in time) reaction diffusion to the Hilbert set-up but it is not so: a cut-off of the form $\varphi(\|x\|_{L^2})$ does not make a polynomial x^n locally Lipschitz in L^2 . Concerning the Hölder term B , there are examples in E which are not even defined as operators on $L^2(0, 1)$, see section 0.1.

Before the more recent works (the present one and the other mentioned above) on pathwise uniqueness for abstract stochastic evolution equations in Hilbert or Banach spaces, there have been several important works on one-dimensional SPDEs of parabolic type driven by space-time white noise, with several levels of generality of the drift term, see [15], [12], [14], [13], [1]. These works remain highly competitive with the abstract ones, and sometimes more general, but conversely the abstract works cover examples not treated there. Concerning reaction-diffusion, some examples are included in these previous works but not in the generality treated here and moreover, the abstract nature of the Hölder term B allows us to treat new examples, like those of section 0.1.

Finally, we want to stress that this paper contains, for the purpose of pathwise uniqueness, a detailed analysis of the Kolmogorov equation associated to the SPDE above. These results may have other applications and also an intrinsic interest for infinite dimensional analysis. The Kolmogorov equation associated to reaction-diffusion equation has been investigated in [4], [5], [6] and related works. In our work here we add new informations. First, an improved analysis of second derivatives is given, needed to control one of the terms which appears in the reformulated evolution equation (one of the main points for the proof of pathwise uniqueness). Second, a vectorial form of the Kolmogorov equation is discussed, again needed in this particular approach to pathwise uniqueness. Third, the classical case of the Kolmogorov equation with reaction diffusion term F has been extended to cover also the Hölder operator B .

0.1 Examples

Let $E = C([0, 1])$, $H = L^2(0, 1)$. We give two examples of maps $B : E \rightarrow E$ which are not well defined as maps from H to H , and are of class

$$B \in C_b^\alpha(E, E).$$

This shows that our theory has more applications than the previous works.

Example 0.1. Given $g \in E$, $\xi_0 \in [0, 1]$, $b \in C_b^\alpha(\mathbb{R}, \mathbb{R})$ such that

$$|b(s) - b(s')| \leq M |s - s'|^\alpha$$

set

$$B(x)(\xi) = b(x(\xi_0))g(\xi), \quad x \in E.$$

Then $B \in C_b^\alpha(E, E)$. Indeed,

$$\begin{aligned} |B(x) - B(x')|_E &= \max_{\xi \in [0, 1]} |b(x(\xi_0))g(\xi) - b(x'(\xi_0))g(\xi)| \\ &= |b(x(\xi_0)) - b(x'(\xi_0))| |g|_E \leq M |x(\xi_0) - x'(\xi_0)|^\alpha |g|_E \leq M \|g\|_E |x - x'|_E^\alpha. \end{aligned}$$

Example 0.2. Given b as above, set

$$B(x)(\xi) = b\left(\max_{s \in [0, \xi]} x(s)\right).$$

Then $B \in C_b^\alpha(E, E)$. Indeed,

$$\begin{aligned} |B(x) - B(x')|_E &= \max_{\xi \in [0, 1]} \left| b\left(\max_{s \in [0, \xi]} x(s)\right) - b\left(\max_{s \in [0, \xi]} x'(s)\right) \right| \\ &\leq M \max_{\xi \in [0, 1]} \left| \max_{s \in [0, \xi]} x(s) - \max_{s \in [0, \xi]} x'(s) \right|^\alpha. \end{aligned}$$

Now, one has

$$\left| \max_{s \in [0, \xi]} x(s) - \max_{s \in [0, \xi]} x'(s) \right| \leq \max_{s \in [0, \xi]} |x(s) - x'(s)| \quad (0.1)$$

Indeed, assume that

$$\max_{s \in [0, \xi]} x(s) \geq \max_{s \in [0, \xi]} x'(s).$$

Let $s_M, s'_M \in [0, \xi]$ be two points such that

$$\max_{s \in [0, \xi]} x(s) = x(s_M), \quad \max_{s \in [0, \xi]} x'(s) = x'(s'_M).$$

We have

$$x'(s'_M) \geq x'(s_M)$$

and thus

$$\begin{aligned} \max_{s \in [0, \xi]} x(s) - \max_{s \in [0, \xi]} x'(s) &= x(s_M) - x'(s'_M) \leq x(s_M) - x'(s_M) \\ &\leq \max_{s \in [0, \xi]} (x(s) - x'(s)) \leq \max_{s \in [0, \xi]} |x(s) - x'(s)|. \end{aligned}$$

We arrive to the same conclusion if $\max_{s \in [0, \xi]} x(s) \leq \max_{s \in [0, \xi]} x'(s)$. Therefore we have proved (0.1). We apply it to the estimates above and get

$$\begin{aligned} |B(x) - B(x')|_E &\leq M \max_{\xi \in [0, 1]} \left(\max_{s \in [0, \xi]} |x(s) - x'(s)| \right)^\alpha \\ &= M \max_{\xi \in [0, 1]} \max_{s \in [0, \xi]} |x(s) - x'(s)|^\alpha = M |x - x'|_E^\alpha. \end{aligned}$$

The proof is complete.

Example 0.3. With minor adjustments the same result holds for

$$B(u)(\xi) = b\left(\max_{s \in [0, \xi]} |u(s)|\right).$$

Remark 0.4. On the contrary, the example

$$B(u)(\xi) = b(u(\xi))$$

is also of class $B \in C_b^\alpha(H, H)$ and thus it is covered by the previous theories. Indeed,

$$\begin{aligned} \|B(u) - B(u')\|_H^2 &= \int_0^1 |b(u(\xi)) - b(u'(\xi))|^2 d\xi \\ &\leq M^2 \int_0^1 |u(\xi) - u'(\xi)|^{2\alpha} d\xi \leq M^2 \left(\int_0^1 |u(\xi) - u'(\xi)|^2 d\xi \right)^\alpha = M^2 \|u - u'\|_H^{2\alpha}. \end{aligned}$$

0.2 Notations

Let X and Y be two separable Banach spaces. In what follows, we shall denote by $B_b(X, Y)$ the Banach space of bounded Borel function $\varphi : X \rightarrow Y$, endowed with the sup-norm

$$\|\varphi\|_{B_b(X, Y)} := \sup_{x \in X} |\varphi(x)|_Y,$$

and by $C_b(X, Y)$ the subspace of uniformly continuous mappings. $\text{Lip}_b(X, Y)$ is the subspace of Lipschitz-continuous mappings, endowed with the norm

$$\|\varphi\|_{\text{Lip}_b(X, Y)} := \|\varphi\|_{C_b(X, Y)} + \sup_{x, y \in E, x \neq y} \frac{|\varphi(x) - \varphi(y)|_Y}{|x - y|_X} =: \|\varphi\|_{C_b(X, Y)} + [\varphi]_{\text{Lip}_b(X, Y)}.$$

For any $\theta \in (0, 1)$, we denote by $C_b^\theta(X, Y)$ the Banach space of all θ -Hölder continuous mappings $\varphi \in C_b(X, Y)$, endowed with the norm

$$\|\varphi\|_{C^\theta(X, Y)} = \|\varphi\|_{C_b(X, Y)} + \sup_{x, y \in X, x \neq y} \frac{|\varphi(x) - \varphi(y)|_Y}{|x - y|_X^\theta}.$$

Finally, for any integer $k \geq 1$, we denote by $C_b^k(X, Y)$ the space of all mappings $\varphi : X \rightarrow Y$ which are k times differentiable, with uniformly continuous and bounded derivatives. $C_b^k(X, Y)$ is a Banach space, endowed with the norm

$$\|\varphi\|_{C_b^k(X, Y)} =: \|\varphi\|_{C_b(X, Y)} + \sum_{j=1}^k \sup_{x \in X} \|D^j \varphi(x)\|_{\mathcal{L}^j(X, Y)}.$$

Spaces $C_b^{\theta+k}(X, Y)$, with $k \in \mathbb{N}$ and $\theta \in (0, 1)$, are defined similarly.

Finally, when $Y = \mathbb{R}$, we shall denote $B_b(X, Y)$ and $C_b^{\theta+k}(X, Y)$, for $\theta \in [0, 1]$ and $k \in \mathbb{N}$, by $B_b(X)$ and $C_b^{\theta+k}(X)$, respectively.

1 The unperturbed reaction-diffusion equation

We are here concerned with the following stochastic reaction–diffusion equation in the Banach space $C([0, 1])$,

$$\begin{cases} dX(t, \xi) = [D_\xi^2 X(t, \xi) + f(\xi, X(t, \xi))]dt + dw(t, \xi), & \xi \in (0, 1), \\ \mathcal{B}X(t, 0) = \mathcal{B}X(t, 1) = 0, & t \geq 0, \\ X(0, \xi) = x(\xi), & \xi \in [0, 1], \end{cases} \quad (1.1)$$

where $b : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a given function, $w(t)$ is a cylindrical Wiener process in $L^2(0, 1)$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, and either $\mathcal{B}u = u$ (Dirichlet boundary condition) or $\mathcal{B}u = u'$ (Neumann boundary condition).

If we denote by A the realization in $C([0, 1])$ of the operator D_ξ^2 , endowed with the boundary condition \mathcal{B} , and if we denote by F the Nemytski operator associated with f , namely

$$F(x)(\xi) = f(\xi, x(\xi)), \quad x \in C([0, 1]), \quad \xi \in [0, 1],$$

then problem (1.1) can be written as the following stochastic differential equation in $C([0, 1])$

$$\begin{cases} dX(t) = [AX(t) + F(X(t))]dt + dw(t), \\ X(0) = x. \end{cases} \quad (1.2)$$

In what follows, we shall denote by H the Hilbert space $L^2(0, 1)$, endowed with the scalar product $\langle \cdot, \cdot \rangle_H$ and the corresponding norm $|\cdot|_H$. With E we shall denote the closure of $D(A)$ in the space $C([0, 1])$, endowed with the uniform norm $|\cdot|_E$ and the duality $\langle \cdot, \cdot \rangle_E$ between E and E^* . Notice that in the case of Dirichlet boundary conditions $\overline{D(A)} = C_0([0, 1])$ and in the case of Neumann boundary conditions $\overline{D(A)} = C([0, 1])$. However, in both cases the semigroup e^{tA} generated by A is strongly continuous and analytic in E . Finally, for any $\epsilon > 0$ we shall denote by E_ϵ the subspace of ϵ -Hölder continuous functions, endowed with the norm

$$|x|_{E_\epsilon} := |x|_E + \sup_{\substack{\xi, \eta \in [0, 1] \\ \xi \neq \eta}} \frac{|x(\xi) - x(\eta)|}{|\xi - \eta|_E^\epsilon}.$$

In what follows, we shall assume that the mapping $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies the following conditions.

Hypothesis 1. *1. For any $\xi \in [0, 1]$, the mapping $f(\xi, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is of class C^3 and there exists an integer $m \geq 0$ such that*

$$\sup_{\substack{\xi \in [0, 1] \\ s \in \mathbb{R}}} \frac{|D_s^j f(\xi, s)|}{1 + |s|^{2m+1-j}} < \infty, \quad j = 0, 1, 2, 3. \quad (1.3)$$

Moreover, the mappings $D_s^j f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are all continuous.

2. We have

$$\sup_{\substack{\xi \in [0,1] \\ s \in \mathbb{R}}} Df(\xi, s) =: \rho < \infty. \quad (1.4)$$

3. If $m \geq 1$, then there exist $\alpha > 0$, $\gamma \geq 0$ and $c \in \mathbb{R}$ such that

$$\sup_{\xi \in [0,1]} (f(\xi, s+h) - f(\xi, s)) h \leq -\alpha h^{2(m+1)} + c(1 + |s|^\gamma).$$

A simple example of a function f fulfilling all conditions in Hypothesis 1 is

$$f(\xi, s) = -\alpha(\xi) s^{2m+1} + \sum_{j=0}^{2m} c_j(\xi) s^j, \quad (\xi, s) \in [0, 1] \times \mathbb{R},$$

for some continuous functions $\alpha, c_j : [0, 1] \rightarrow \mathbb{R}$, with

$$\inf_{\xi \in [0,1]} \alpha(\xi) =: \alpha_0 > 0.$$

Definition 1.1. Let $x \in E$. We say that an adapted process $X(\cdot, x)$ is a *mild solution* of problem (1.1) if $X(t, x) \in E$, for all $t \geq 0$, and fulfills the integral equation

$$X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A} F(X(s)) ds + W_A(t), \quad t \geq 0, \quad (1.5)$$

where $W_A(t)$ is the stochastic convolution

$$W_A(t) = \int_0^t e^{(t-s)A} dw(s), \quad t \geq 0.$$

In [4, Proposition 6.2.2] is proved that, for any $x \in E$, problem (1.1) admits a unique adapted *mild solution* $X(\cdot, x) \in L^p(\Omega; C([0, T]; E))$, for any $T > 0$ and $p \geq 1$, such that for any $t \in [0, T]$

$$\sup_{s \in [0, t]} |X(s, x)|_E \leq \Lambda(t) (1 + |x|_E), \quad \mathbb{P} - \text{a.s.} \quad (1.6)$$

for some random variable $\Lambda(t)$ such that

$$\mathbb{E} \Lambda(t)^p < \infty,$$

for any $p \geq 1$. Moreover, in [4, Theorem 6.2.3] is proved that for any $t > 0$

$$\sup_{x \in E} |X(t, x)|_E \leq \Gamma(t) t^{-\frac{1}{2m}}, \quad \mathbb{P} - \text{a.s.} \quad (1.7)$$

for some random variable $\Gamma(t)$, increasing with respect to t , such that

$$\mathbb{E} \Gamma(t)^p < \infty,$$

for any $p \geq 1$ and $t \geq 0$.

Notice that there exists $\epsilon_0 > 0$ such that for any $x \in E$

$$X(t, x) \in E_{\epsilon_0}, \quad t > 0, \quad \mathbb{P} - \text{a.s.} \quad (1.8)$$

and the mapping $x \in E \rightarrow X(t, x) \in E_{\epsilon_0}$ is continuous, \mathbb{P} -a.s.

Moreover, in [4, Proposition 7.1.2] it has been proved that for any $x \in H$ there exists a unique *generalized solution* $X(\cdot, x) \in L^p(\Omega; C([0, T]; H))$, for any $p \geq 1$ and $T > 0$. This means that for any sequence $\{x_n\}_{n \in \mathbb{N}} \subset E$ converging to x in H , the sequence $\{X(\cdot, x_n)\}_{n \in \mathbb{N}}$ converges to $X(\cdot, x)$ in $C([0, T]; H)$, \mathbb{P} -a.s. as $n \rightarrow \infty$. Furthermore, estimates analogous to (1.6) and (1.7) hold in H . Namely,

$$\sup_{s \in [0, t]} |X(s, x)|_H \leq \Lambda(t) (1 + |x|_H), \quad \mathbb{P} - \text{a.s.} \quad (1.9)$$

and

$$\sup_{x \in H} |X(t, x)|_H \leq \Gamma(t) t^{-\frac{1}{2m}}, \quad \mathbb{P} - \text{a.s.} \quad (1.10)$$

for suitable random variables $\Lambda(t)$ and $\Gamma(t)$ as above.

In [4, Chapter 6] the regularity of the mapping

$$x \in E \mapsto X(t, x) \in C([0, T]; L^p(\Omega, E)),$$

has been studied and in Theorem 6.3.3 it has been proved that, as f is assumed to be of class C^3 , such a mapping is three times differentiable and the derivatives satisfy

$$\sup_{\substack{x \in E \\ t \in [0, T]}} |D_x^j X(t, x)(h_1, \dots, h_r)|_E \leq \Lambda_j(T) |h_1|_E \cdots |h_r|_E, \quad (1.11)$$

for any $r = 1, 2, 3$, $T > 0$ and $h_1, \dots, h_r \in E$ and for some random variables $\Lambda_j(T)$ having finite moments of any order.

The regularity of the mapping

$$x \in H \mapsto X(t, x) \in C([0, T]; L^p(\Omega, H)),$$

has not been investigated, but in [4, Proposition 7.2.1] it has been proved that for any $x, h \in H$ there exists a process $v(\cdot, x, h)$ such that for any two sequences $\{x_n\}_{n \in \mathbb{N}}$ and $\{h_n\}_{n \in \mathbb{N}}$, converging in H to x and h , respectively, the sequence $\{D_x X(\cdot, x_n) h_n\}_{n \in \mathbb{N}}$ converges to $v(\cdot, x, h)$ in $C([0, T]; H)$, \mathbb{P} -a.s.

Now, for any $x \in E$, $h \in H$ and $s \geq 0$, let us consider the problem

$$\eta'(t) = A\eta(t) + F'(X(t, x))\eta(t), \quad \eta(s) = h, \quad t \geq s, \quad \omega \in \Omega. \quad (1.12)$$

This is a random equation, whose solution is denoted by $\eta(t; s, x, h)$, and it defines the following random evolution operator

$$[U_{t,s}^x(\omega)]h = \eta(t; s, x, h)(\omega). \quad (1.13)$$

In view of Hypothesis 1, it is immediate to check that $U_{t,s}^x$ satisfies the following properties (the proof is left to the reader).

Lemma 1.2. 1. *There exists a kernel $K_{t,s}^x : \Omega \times [0, 1] \times [0, 1] \rightarrow \mathbb{R}^+$ such that for any $x \in E$, $h \in H$ and $0 \leq s \leq t$*

$$U_{t,s}^x(\omega)h(\xi) = \int_0^1 K_{t,s}^x(\omega, \xi, \theta)h(\theta) d\theta. \quad (1.14)$$

2. *For any $(\xi, \theta) \in [0, 1] \times [0, 1]$ and $x \in E$, we have*

$$0 \leq K_{t,s}^x(\omega, \xi, \theta) \leq K_{t-s}(\xi, \theta) e^{\rho t} \leq (4\pi t)^{-\frac{1}{2}} e^{\rho t}, \quad 0 \leq s \leq t, \quad \mathbb{P} - a.s. \quad (1.15)$$

where ρ is the constant introduced in (1.4) and $K_t(\xi, \theta)$ is the kernel associated with the operator A .

3. *The evolution operator $U_{t,s}^x$ is ultracontractive and for any $1 \leq q \leq p$*

$$|U_{t,s}^x(\omega)h|_p \leq c((t-s) \wedge 1)^{-\frac{p-q}{2pq}} |h|_q, \quad t > s, \quad \mathbb{P} - a.s. \quad (1.16)$$

As a consequence of the previous Lemma, the following fact holds.

Lemma 1.3. *We have*

$$\sup_{x \in E} \sum_{i=1}^{\infty} |D_x X(t, x) e_i|_H^2 \leq c e^{2\rho t} t^{-\frac{1}{2}}, \quad t > 0, \quad \mathbb{P} - a.s. \quad (1.17)$$

for some constant $c > 0$. Moreover, the sum converges uniformly with respect to $x \in E$.

Proof. We have $D_x X(t, x) e_i = U_{t,0}^x e_i$, hence, due to (1.14), we have

$$\sum_{i=1}^{\infty} |D_x X(t, x) e_i(\xi)|^2 = \sum_{i=1}^{\infty} \left| \langle K_{t,0}^x(\xi, \cdot), e_i \rangle_H \right|^2 = |K_{t,0}^x(\xi, \cdot)|_H^2 \leq |K_t(\xi, \cdot)|_H^2 e^{2\rho t}.$$

This implies that for any $t > 0$

$$\sum_{i=1}^{\infty} |D_x X(t, x) e_i|_H^2 \leq \int_0^1 |K_t(\xi, \cdot)|_H^2 d\xi e^{2\rho t} \leq c e^{2\rho t} t^{-\frac{1}{2}}.$$

□

Remark 1.4. Due to (1.16), for any $1 \leq p \leq q \leq \infty$ we have

$$|D_x X(t, x) v|_p \leq c(t \wedge 1)^{-\frac{p-q}{2pq}} |v|_q. \quad (1.18)$$

In particular, if $x, y \in E$, we have

$$|X(t, x) - X(t, y)|_H \leq \int_0^1 |D_x X(t, \theta x + (1-\theta)y)(x-y)|_H d\theta \leq c(t)|x-y|_H.$$

Recalling how the generalized solution $X(t, x)$ has been constructed in H , this implies that for any $x, y \in H$ and $t \geq 0$

$$|X(t, x) - X(t, y)|_H \leq c(t)|x-y|_H. \quad (1.19)$$

Lemma 1.5. *For any $x, h \in E$ and $t \geq 0$, we have*

$$\sum_{i=1}^{\infty} |D_x^2 X(t, x)(e_i, e_i)|_H \leq \kappa(t) (1 + |x|_E^{2m-1}) (t \wedge 1)^{\frac{1}{4}}, \quad \mathbb{P} - a.s. \quad (1.20)$$

for some positive random variable $\kappa(t)$, increasing with respect to $t \geq 0$, having all moments finite. Moreover, the sum converges uniformly with respect to $x \in B_E(R)$, for any $R > 0$.

Proof. We have

$$D_x^2 X(t, x)(e_i, e_i) = \int_0^t U_{t,s}^x F''(X(s, x))(D_x X(s, x)e_i, D_x X(s, x)e_i) ds.$$

Then, due to (1.6), (1.16) and (1.18), we have

$$\begin{aligned} |D_x^2 X(t, x)(e_i, e_i)|_H &\leq c \int_0^t ((t-s) \wedge 1)^{-\frac{1}{4}} |F''(X(s, x))|_E |D_x X(s, x)e_i|_H^2 ds \\ &\leq c e^{\rho t} \int_0^t ((t-s) \wedge 1)^{-\frac{1}{4}} (1 + |X(s, x)|_E^{2m-1}) |D_x X(s, x)e_i|_H^2 ds \\ &\leq c e^{\rho t} \Lambda(t)^{2m-1} (1 + |x|_E^{2m-1}) \int_0^t ((t-s) \wedge 1)^{-\frac{1}{4}} |D_x X(s, x)e_i|_H^2 ds. \end{aligned}$$

Thanks to (1.17), this implies

$$\sum_{i=1}^{\infty} |D_x^2 X(s, x)(e_i, e_i)|_H ds \leq c e^{3\rho t} \Lambda(t)_H^{2m-1} (1 + |x|_E^{2m-1}) \int_0^t ((t-s) \wedge 1)^{-\frac{1}{4}} s^{-\frac{1}{2}} ds,$$

and (1.20) follows. □

Remark 1.6. Let $J_n = nR(n, A)$. Then, from the proof above, we have also that

$$\sum_{i=1}^{\infty} |D_x^2 X(t, J_n x)(J_n e_i, J_n e_i)|_H \leq \kappa(t) (1 + |x|_E^{2m-1}). \quad (1.21)$$

Notice that the series converges uniformly with respect to $n \in \mathbb{N}$ and $x \in B_E(R)$.

Lemma 1.7. *For any $x, y \in E$ and $h \in H$ and $t > 0$, we have*

$$|D_x X(t, x)h - D_x X(t, y)h|_H \leq \kappa(t) |x - y|_E |h|_H (t \wedge 1)^{\frac{1}{2m}}, \quad \mathbb{P} - a.s. \quad (1.22)$$

where $\kappa(t)$ is a random variable, increasing with respect to t , and having finite moments of any order.

Proof. If we define $\rho(t) := D_x X(t, x)h - D_x X(t, y)h$, we have

$$\rho'(t) = A\rho(t) + F'(X(t, x))\rho(t) + [F'(X(t, x)) - F'(X(t, y))] D_x X(t, y)h, \quad \rho(0) = 0,$$

and then,

$$\rho(t) = \int_0^t U_{t,s}^x [F'(X(s, x)) - F'(X(s, y))] D_x X(s, y)h \, ds.$$

According to (1.7) and (1.16), this yields

$$\begin{aligned} |\rho(t)|_H &\leq c \int_0^t |[F'(X(s, x)) - F'(X(s, y))] D_x X(s, y)h|_H \, ds \\ &\leq c \int_0^t (1 + |X(t, x)|_E^{2m-1} + |X(t, y)|_E^{2m-1}) |X(s, x) - X(s, y)|_E |D_x X(s, y)h|_H \, ds \\ &\leq \Lambda(t) \int_0^t \left(1 + s^{-1+\frac{1}{2m}}\right) |X(s, x) - X(s, y)|_E |D_x X(s, y)h|_H \, ds \end{aligned}$$

and due to (1.18) this allows to conclude. □

Remark 1.8. 1. Since

$$|U_{t,s}^x h|_E \leq c((t-s) \wedge 1)^{-\frac{1}{4}} |h|_H,$$

from the proof above, we easily see that for any $x, y \in E$ and $h \in H$

$$|D_x X(t, x)h - D_x X(t, y)h|_E \leq \kappa(t) |x - y|_E |h|_H (t \wedge 1)^{-\frac{1}{4} + \frac{1}{2m}}, \quad \mathbb{P} - \text{a.s.} \quad (1.23)$$

for some random variable $\kappa(t)$ as in Lemma 1.7

2. Let $v(\cdot, x, h)$ be the process defined above as

$$\lim_{n \rightarrow \infty} D_x X(t, x_n)h_n, \quad \text{in } H,$$

for any two sequences $\{x_n\}_{n \in \mathbb{N}}$ and $\{h_n\}_{n \in \mathbb{N}}$, converging in H to x and h , respectively.

Then, as above for $D_x X(t, x)h$, we have that for any $x, y, h \in H$ and $t > 0$

$$|v(t, x, h) - v(t, y, h)|_H \leq \kappa(t) |x - y|_H |h|_H (t \wedge 1)^{-\frac{1}{4} + \frac{1}{2m}}, \quad (1.24)$$

where $\kappa(t)$ is a random variable, increasing with respect to t , and having finite moments of any order.

2 The unperturbed semigroup

In what follows, we shall denote by P_t the Markov transition semigroup associated with equation (1.1) in E . Namely

$$P_t \varphi(x) = \mathbb{E} \varphi(X(t, x)), \quad x \in E, \quad (2.1)$$

for any $\varphi \in B_b(E)$ and $t \geq 0$, where $X(t, x)$ is the unique mild solution of equation (1.1). Moreover, we shall denote by P_t^H the transition semigroup associated with equation (1.1) in H . Namely

$$P_t^H \varphi(x) = \mathbb{E} \varphi(X(t, x)), \quad x \in H, \quad (2.2)$$

for any $\varphi \in B_b(H)$, where $X(t, x)$ is the unique generalized solution of equation (1.1) in H . Notice that since E is a Borel subset of H , if $x \in E$ and $\varphi \in B_b(E)$, then

$$P_t \varphi(x) = P_t^H \varphi(x), \quad t \geq 0. \quad (2.3)$$

For this reason, in what follows we may not distinguish between P_t and P_t^H when it is not necessary.

In [4, Theorem 6.5.1]) it is proved that the semigroup P_t has a smoothing effect and, in spite of the polynomial growth of f , uniform bounds are satisfied by the derivatives of $P_t \varphi$. Actually, we have the following result.

Proposition 2.1. *For any $\varphi \in B_b(E)$ and $t > 0$, we have that $P_t \varphi \in C_b^3(E)$ and for any $0 \leq i \leq j \leq 3$*

$$\|P_t \varphi\|_{C_b^j(E)} \leq c_j (t \wedge 1)^{-\frac{j-i}{2}} \|\varphi\|_{C_b^i(E)}. \quad (2.4)$$

Moreover, it holds

$$\langle h, D(P_t \varphi)(x) \rangle_E = \frac{1}{t} \mathbb{E} \varphi(X(t, x)) \int_0^t \langle D_x X(s, x) h, dw(s) \rangle_H. \quad (2.5)$$

As a consequence of (2.5) and (1.18), we have

$$\begin{aligned} |\langle h, D(P_t \varphi)(x) \rangle_E| &\leq \frac{1}{t} \|\varphi\|_{C_b(E)} \left(\mathbb{E} \int_0^t |D_x X(s, x) h|_H^2 ds \right)^{\frac{1}{2}} \\ &\leq c(t) (t \wedge 1)^{-\frac{1}{2}} \|\varphi\|_{C_b(E)} |h|_H, \end{aligned}$$

so that $D(P_t \varphi)(x)$ can be extended to a linear functional on H , for any $x \in E$ and $t > 0$, and

$$|D(P_t \varphi)|_{C_b(E, H)} \leq c(t) (t \wedge 1)^{-\frac{1}{2}} \|\varphi\|_{C_b(E)}. \quad (2.6)$$

In fact, the mapping $D(P_t \varphi) : E \rightarrow H$ is Lipschitz-continuous, as shown in next lemma.

Lemma 2.2. *For any $\varphi \in C_b(E)$, $x \in E$ and $t > 0$ we have that $D(P_t \varphi)(x) \in H$ and for $j = 0, 1$*

$$|D(P_t \varphi)(x) - D(P_t \varphi)(y)|_H \leq c(t) (t \wedge 1)^{-\frac{2-j}{2}} \|\varphi\|_{C_b^j(E)} |x - y|_E, \quad x, y \in E. \quad (2.7)$$

Proof. Assume $\varphi \in C_b^1(E)$ and fix $x, y, h \in E$ and $t > 0$. Then,

$$\begin{aligned} \langle h, (D(P_t \varphi)(x) - D(P_t \varphi)(y)) \rangle_E &= \frac{1}{t} \mathbb{E} [\varphi(X(t, x)) - \varphi(X(t, y))] \int_0^t \langle D_x X(s, x) h, dw(s) \rangle_H \\ &\quad + \frac{1}{t} \mathbb{E} \varphi(X(t, y)) \int_0^t \langle D_x X(s, x) h - D_x X(s, y) h, dw(s) \rangle_H. \end{aligned}$$

Therefore, thanks to (1.18) and (1.22), we get

$$\begin{aligned}
|\langle h, (D(P_t\varphi)(x) - D(P_t\varphi)(y)) \rangle_E| &\leq \frac{c(t)}{t} \|\varphi\|_{C_b^1(E)} |x - y|_E \left(\mathbb{E} \int_0^t |D_x X(s, x) h|_H^2 ds \right)^{\frac{1}{2}} \\
&+ \frac{1}{t} \|\varphi\|_{C_b(E)} \left(\mathbb{E} \int_0^t |D_x X(s, x) h - D_x X(s, y) h|_H^2 ds \right)^{\frac{1}{2}} \\
&\leq c(t)(t \wedge 1)^{-\frac{1}{2}} \|\varphi\|_{C_b^1(E)} |h|_H |x - y|_E + c(t)(t \wedge 1)^{\frac{1}{2m}} \|\varphi\|_{C_b(E)} |h|_H |x - y|_E,
\end{aligned}$$

and this implies (2.7) for $j = 1$. The case $j = 0$ follows from (2.4) and the semigroup law. \square

Next, we recall that in [6, Section 3], by using suitable interpolation estimates for real-valued functions defined in the Banach space E , we have proved the following result.

Proposition 2.3. *For any $\theta \in (0, 1)$ and $j = 2, 3$, there exists $c_{\theta,j} > 0$ such that for all $\varphi \in C_b^\theta(E)$ and all $t > 0$*

$$\|P_t\varphi\|_{C_b^j(E)} \leq c_{\theta,j}(t \wedge 1)^{-\frac{j-\theta}{2}} \|\varphi\|_{C_b^\theta(E)}. \quad (2.8)$$

As a consequence of (2.7), Proposition 2.3 and the semigroup law imply that for any $\varphi \in C_b^\theta(E)$, with $\theta \in [0, 1]$, and for any $x, y \in E$ and $t > 0$

$$|D(P_t\varphi)(x) - D(P_t\varphi)(y)|_H \leq c(t)(t \wedge 1)^{-\frac{2-\theta}{2}} \|\varphi\|_{C_b^\theta(E)} |x - y|_E, \quad x, y \in E. \quad (2.9)$$

In [4, Theorem 7.3.1] we have also shown that for any $t > 0$ the semigroup P_t^H maps $B_b(H)$ into $C_b^1(H)$ and

$$\langle h, D(P_t^H\varphi)(x) \rangle_H = \frac{1}{t} \mathbb{E} \varphi(X(t, x)) \int_0^t \langle v(s, x, h), dw(s) \rangle_H, \quad (2.10)$$

where $v(\cdot, x, h)$ is the process defined in the previous section as the limit of the derivatives $D_x X(t, x_n) h_n$, where $\{x_n\}_{n \in \mathbb{N}}$ and $\{h_n\}_{n \in \mathbb{N}}$ are two sequences in E converging respectively to x and h in H . In particular, we have shown that

$$\|P_t^H\varphi\|_{C_b^1(H)} \leq c(t \wedge 1)^{-\frac{1}{2}} \|\varphi\|_{C_b(H)}. \quad (2.11)$$

Thanks to (1.19), we have that $P_t^H : C_b^\alpha(H) \rightarrow C_b^\alpha(H)$, for any $\alpha \in [0, 1]$, and $P_t^H : \text{Lip}_b(H) \rightarrow \text{Lip}_b(H)$, with

$$\|P_t^H\varphi\|_{C_b^\alpha(H)} \leq c(t) \|\varphi\|_{C_b^\alpha(H)}, \quad \|P_t^H\varphi\|_{\text{Lip}_b(H)} \leq c(t) \|\varphi\|_{\text{Lip}_b(H)}.$$

Therefore, by interpolation, we have that $P_t^H : C_b^\alpha(H) \rightarrow C_b^\beta(H)$, for any $0 \leq \alpha \leq \beta \leq 1$, and

$$\|P_t^H\varphi\|_{C_b^\beta(H)} \leq c(t) (t \wedge 1)^{-\frac{\beta-\alpha}{2}} \|\varphi\|_{C_b^\alpha(H)}, \quad t > 0. \quad (2.12)$$

Lemma 2.4. *Let $0 \leq \alpha < \beta < 1$ and let $\varphi \in C_b^\alpha(H)$. Then $P_t^H \varphi \in C_b^{1+\beta}(H)$, for any $t > 0$, and*

$$\|P_t^H \varphi\|_{C_b^{1+\beta}(H)} \leq c(t) (t \wedge 1)^{-(\delta + \frac{\beta - \alpha}{2})} \|\varphi\|_{C_b^\alpha(H)}, \quad (2.13)$$

where

$$\delta = \begin{cases} \frac{1}{2}, & \text{if } m = 1, \\ \frac{3}{4} - \frac{1}{2m}, & \text{if } m > 1. \end{cases} \quad (2.14)$$

Proof. Assume $\varphi \in C_b(H)$. Then,

$$\begin{aligned} \langle h, D(P_t^H \varphi)(x) - D(P_t^H \varphi)(y) \rangle_H &= \frac{1}{t} \mathbb{E} [\varphi(X(t, x)) - \varphi(X(t, y))] \int_0^t \langle v(s, x, h), dw(s) \rangle_H \\ &+ \frac{1}{t} \mathbb{E} \varphi(X(t, y)) \int_0^t \langle v(s, x, h) - v(s, y, h), dw(s) \rangle_H \end{aligned}$$

Therefore, thanks to (1.19) and (1.24), we get

$$\begin{aligned} |\langle h, D(P_t^H \varphi)(x) - D(P_t^H \varphi)(y) \rangle_H| &\leq \frac{c(t)}{t} \|\varphi\|_{C_b^\beta(H)} \left(\mathbb{E} \int_0^t |v(s, x, h)|_H^2 ds \right)^{\frac{1}{2}} |x - y|_H^\beta \\ &+ \frac{1}{t} \|\varphi\|_{C_b(H)} \left(\mathbb{E} \int_0^t |v(s, x, h) - v(s, y, h)|_H^2 ds \right)^{\frac{1}{2}} \\ &\leq c(t) (t \wedge 1)^{-\frac{1}{2}} \|\varphi\|_{C_b^\beta(H)} |x - y|_H^\beta |h|_H + c(t) (t \wedge 1)^{-\frac{3}{4} + \frac{1}{2m}} |h|_H. \end{aligned}$$

By the semigroup law, this implies that

$$|D(P_t^H \varphi)(x) - D(P_t^H \varphi)(y)|_H \leq c(t) (t \wedge 1)^{-\delta} \|P_{t/2}^H \varphi\|_{C_b^\beta(H)} |x - y|_H^\beta,$$

so that (2.13) follows from (2.12). □

By proceeding as in [3] (see also [4, Appendix B]), we introduce the generator of P_t . For any $\lambda > 0$ and $\varphi \in C_b(E)$ we define

$$F(\lambda) \varphi(x) = \int_0^\infty e^{-\lambda t} P_t \varphi(x) dt, \quad x \in E. \quad (2.15)$$

As proved e.g. in [4, Proposition B.1.4], there exists a unique m -dissipative operator \mathcal{L} in $C_b(E)$ such that

$$R(\lambda, \mathcal{L}) = (\lambda - \mathcal{L})^{-1} = F(\lambda), \quad \lambda > 0.$$

$\mathcal{L} : D(\mathcal{L}) \subseteq C_b(E) \rightarrow C_b(E)$ is the *weak infinitesimal generator* of P_t . We would like to recall that, as proved in [3] (see also [17] and [4]), if $\varphi \in D(\mathcal{L})$, then

$$\lim_{t \rightarrow 0} \Delta_t \varphi(x) = \mathcal{L} \varphi(x), \quad x \in E,$$

and

$$\sup_{t \in (0,1]} \|\Delta_t \varphi\|_{C_b(E)} < \infty,$$

where

$$\Delta_\epsilon = \frac{1}{\epsilon} (P_\epsilon - I), \quad \epsilon \in (0, 1].$$

Moreover, for any $\varphi \in D(\mathcal{L})$ and $t \geq 0$, we have $P_t \varphi \in D(\mathcal{L})$ and

$$\mathcal{L}P_t \varphi = P_t \mathcal{L} \varphi.$$

The mapping, $t \mapsto P_t \varphi(x)$ is differentiable, and

$$\frac{d}{dt} P_t \varphi(x) = \mathcal{L}P_t \varphi(x) = P_t \mathcal{L} \varphi(x), \quad x \in E.$$

First of all, we notice that

$$\|R(\lambda, \mathcal{L})\varphi\|_{C_b(E)} \leq \frac{1}{\lambda} \|\varphi\|_{C_b(E)}, \quad \varphi \in C_b(E).$$

Moreover, as due to (2.4) we have

$$\|P_t \varphi\|_{C_b^1(E)} \leq c(t \wedge 1)^{-\frac{1}{2}} \|\varphi\|_{C_b(E)},$$

we immediately have that $D(\mathcal{L}) \subset C_b^1(E)$ and

$$\sup_{x \in E} \|D(R(\lambda, \mathcal{L})\varphi)(x)\| \leq \frac{c}{\sqrt{\lambda}} \|\varphi\|_{C_b(E)}. \quad (2.16)$$

Notice that, as a consequence of (2.9), if $\varphi \in C_b^\theta(E)$, with $\theta > 0$, we have that $D(R(\lambda, \mathcal{L})\varphi) : E \rightarrow H$ is well defined, and

$$|D(R(\lambda, \mathcal{L})\varphi)(x) - D(R(\lambda, \mathcal{L})\varphi)(y)| \leq c\lambda^{-\frac{\theta}{2}} \|\varphi\|_{C_b^\theta(E)}, \quad x, y \in E. \quad (2.17)$$

As for P_t and \mathcal{L} , we can also introduce the weak generator \mathcal{L}^H of the semigroup P_t^H . Due to (2.3), for any $\lambda > 0$ and $\varphi \in C_b(H)$ we have

$$R(\lambda, \mathcal{L})\varphi(x) = R(\lambda, \mathcal{L}^H)\varphi(x), \quad x \in E. \quad (2.18)$$

Now, for any $\lambda > 0$ and $\psi \in C_b(E)$, we consider the elliptic equation

$$\lambda\varphi - \mathcal{L}\varphi = \psi. \quad (2.19)$$

As the resolvent set of \mathcal{L} contains $(0, +\infty)$, we have that equation (2.19) admits a unique solution in $C_b(E)$, which is given by $\varphi = R(\lambda, \mathcal{L})\psi$.

In [6] we have proved that in fact Schauder estimates are satisfied by the solution of equation (2.19).

Theorem 2.5. Let $\psi \in C_b^\theta(E)$, with $\theta \in (0, 1)$, and let $\varphi = R(\lambda, \mathcal{L})\psi$, with $\lambda > 0$. Then we have $\varphi \in C_b^{2+\theta}(E)$ and there exists $c > 0$ (independent of ψ) such that

$$\|\varphi\|_{C_b^{2+\theta}(E)} \leq c \|\psi\|_{C_b^\theta(E)}. \quad (2.20)$$

Notice that, in view of Lemma 2.4, we have

$$\psi \in C_b^\alpha(H) \implies \varphi = R(\lambda, \mathcal{L})\psi \in C_b^{1+\beta}(H), \quad (2.21)$$

for any $\beta < 2(1 - \delta) + \alpha$, where δ is the constant defined in (2.14).

Next, we show that under a suitable condition on ψ a *trace property* is satisfied by $D^2\varphi(x)$.

Theorem 2.6. For any $x \in E$ and $\psi \in C_b^\theta(H)$, with $\theta > 1/2$, the series

$$\sum_{i=1}^{\infty} D^2\varphi(x)(e_i, e_i)$$

is convergent and

$$\left| \sum_{i=1}^{\infty} D^2\varphi(x)(e_i, e_i) \right| \leq c \lambda^{\frac{\theta-2}{4}} (1 + |x|_E^{2m-1}) \|\psi\|_{C_b^\theta(H)}. \quad (2.22)$$

Moreover, the convergence is uniform for $x \in B_E(R)$, for any $R > 0$.

Proof. Assume first that $\psi \in C_b^1(H)$. If we differentiate in

$$\langle h, D(P_t\psi)(x) \rangle_E = \frac{1}{t} \mathbb{E} \psi(X(t, x)) \int_0^t \langle D_x X(s, x) h, dw(s) \rangle_H,$$

along the direction $k \in E$, we get

$$\begin{aligned} D^2(P_t\psi)(x)(h, k) &= \frac{1}{t} \mathbb{E} \langle D_x X(t, x) k, D\psi(X(t, x)) \rangle_E \int_0^t \langle D_x X(s, x) h, dw(s) \rangle_H \\ &+ \frac{1}{t} \mathbb{E} \psi(X(t, x)) \int_0^t \langle D_x^2 X(s, x)(h, k), dw(s) \rangle_H. \end{aligned}$$

This means that for any $n, p \in \mathbb{N}$

$$\begin{aligned} &\sum_{i=n}^{n+p} D^2(P_t\psi)(x)(e_i, e_i) \\ &= \frac{1}{t} \mathbb{E} \left\langle D_x X(t, x) \left(\sum_{i=n}^{n+p} \int_0^t \langle D_x X(s, x) e_i, dw(s) \rangle_H e_i \right), D\psi(X(t, x)) \right\rangle_H \\ &+ \frac{1}{t} \mathbb{E} \psi(X(t, x)) \int_0^t \left\langle \sum_{i=n}^{n+p} D_x^2 X(s, x)(e_i, e_i), dw(s) \right\rangle_H =: I_{1,p}^n(t) + I_{2,p}^n(t). \end{aligned}$$

Now, according to (1.18) and (2.6), we have

$$\begin{aligned} |I_{1,p}^n(t)| &\leq \frac{c}{t} \|\psi\|_{C_b^1(H)} \left(\mathbb{E} \left| \sum_{i=n}^{n+p} \int_0^t \langle D_x X(s, x) e_i, dw(s) \rangle_H e_i \right|_H^2 \right)^{\frac{1}{2}} \\ &= \frac{c}{t} \|\psi\|_{C_b^1(H)} \left(\mathbb{E} \int_0^t \sum_{i=n}^{n+p} |D_x X(s, x) e_i|_H^2 ds \right)^{\frac{1}{2}} \end{aligned}$$

and then, due to (1.17) we can conclude that for any $t > 0$ and $p \geq 0$

$$\lim_{n \rightarrow \infty} I_{1,p}^n(t) = 0. \quad (2.23)$$

Moreover, for any $n \in \mathbb{N}$

$$|I_{1,p}^n(t)| \leq c(t) (t \wedge 1)^{-\frac{3}{4}} \|\psi\|_{C_b^1(H)}. \quad (2.24)$$

Next, according to (1.11) we have

$$\begin{aligned} |I_{2,p}^n(t)| &\leq \frac{1}{t} \|\psi\|_{C_b(H)} \left(\int_0^t \mathbb{E} \left| \sum_{i=n}^{n+p} D_x^2 X(s, x)(e_i, e_i) \right|_H^2 ds \right)^{\frac{1}{2}} \\ &\leq \frac{c(t)}{t} \|\psi\|_{C_b(H)} \left(\int_0^t \mathbb{E} \left(\sum_{i=n}^{n+p} |D_x^2 X(s, x)(e_i, e_i)|_H \right)^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Then, as a consequence of (1.20), we get

$$\lim_{n \rightarrow \infty} I_{2,p}^n(t) = 0, \quad (2.25)$$

and for any $n \in \mathbb{N}$

$$|I_{2,p}^n(t)| \leq c(t) (t \wedge 1)^{-\frac{1}{4}} (1 + |x|_E^{2m-1}) \|\psi\|_{C_b(H)}. \quad (2.26)$$

Therefore, as $P_t \psi = P_{t/2}(P_{t/2} \psi)$ and $P_{t/2} \psi \in C_b^1(H)$, (2.23) and (2.25) imply that for any $p \geq 1$

$$\lim_{n \rightarrow \infty} \sum_{i=n}^{n+p} D^2(P_t \psi)(x)(e_i, e_i) = 0. \quad (2.27)$$

Moreover, according to (2.12), (2.24) and (2.26), for any $n \in \mathbb{N}$ we have

$$\left| \sum_{i=1}^n D^2(P_t \psi)(x)(e_i, e_i) \right| \leq c(t) (t \wedge 1)^{-\frac{3}{4} - \frac{1-\theta}{2}} \|\psi\|_{C_b^\theta(H)}. \quad (2.28)$$

Now, as

$$\sum_{i=n}^{n+p} D^2 \varphi(x)(e_i, e_i) = \int_0^\infty e^{-\lambda t} \sum_{i=n}^{n+p} D^2(P_t \psi)(x)(e_i, e_i) dt,$$

from (2.27) and (2.28) we can conclude that if $\theta > 1/2$ then the series $\sum_{i=1}^{\infty} D^2\varphi(x)(e_i, e_i)$ is convergent and (2.22) holds.

The uniformity of the convergence with respect to $x \in B_R(E)$ is a consequence of the uniformity of the convergence in the series in Lemma 1.3 and Lemma 1.5. \square

Remark 2.7. In view of Remark 1.6, if $J_n = nR(n, A)$, then we immediately have that the series

$$\sum_{i=1}^{\infty} D^2\varphi(J_n x)(J_n e_i, J_n e_i),$$

is uniformly convergent, with respect to $x \in B_R(E)$ and $n \in \mathbb{N}$.

3 The vectorial unperturbed semigroup

If $\Phi \in C_b^j(E, E)$, for some positive integer j , we have $D^i\Phi(x)(f_1, \dots, f_i) \in E$, for any $x_1, \dots, x_i \in E$ and any integer $i \leq j$. Moreover, if for $v \in E^*$ we denote

$$\varphi_v(x) := \langle \Phi(x), v \rangle_E, \quad x \in E,$$

we have that $\varphi_v \in C_b^j(E)$, and

$$\langle D^i\Phi(x)(x_1, \dots, x_i), v \rangle_E = D^i\varphi_v(x)(x_1, \dots, x_i), \quad (3.1)$$

so that

$$\|D^i\Phi(x)\|_{\mathcal{L}^i(E, E)} = \sup_{|v|_{E^*} \leq 1} |D^i\varphi_v(x)|_{\mathcal{L}^i(E)}, \quad x \in E. \quad (3.2)$$

Now, for any $\Phi \in C_b(E, E)$, we define

$$\widehat{P}_t\Phi(x) = \mathbb{E}\Phi(X(t, x)), \quad x \in E, \quad t \geq 0.$$

Clearly \widehat{P}_t maps $C_b(E, E)$ into itself and for any $v \in E^*$

$$\langle \widehat{P}_t\Phi(x), v \rangle_E = P_t\varphi_v(x), \quad x \in E, \quad t \geq 0. \quad (3.3)$$

Moreover, it is possible to adapt the arguments used in [4, Theorem 6.5.1] and prove that for any $t > 0$

$$\widehat{P}_t : C_b(E, E) \rightarrow C_b^3(E, E).$$

This implies the following result.

Proposition 3.1. *For any $0 \leq i \leq j \leq 3$ and $t > 0$*

$$\|\widehat{P}_t\Phi\|_{C_b^j(E, E)} \leq c(t \wedge 1)^{-\frac{j-i}{2}} \|\Phi\|_{C_b^i(E, E)}. \quad (3.4)$$

Proof. According to (3.2) and (3.3), we have

$$\begin{aligned} \sup_{x \in E} |D^j(\widehat{P}_t \Phi)(x)|_{\mathcal{L}^j(E)} &= \sup_{|v|_{E^*} \leq 1} \sup_{x \in E} |D^j(\langle \widehat{P}_t \Phi, v \rangle_E)(x)|_{\mathcal{L}^j(E, \mathbb{R})} \\ &= \sup_{|v|_{E^*} \leq 1} \sup_{x \in E} |D^j(P_t \varphi_v)(x)|_{\mathcal{L}^j(E, \mathbb{R})}, \end{aligned}$$

Then, as

$$\sup_{x \in E} |D^i \varphi_v(x)|_{\mathcal{L}^i(E, \mathbb{R})} \leq |v|_{E^*} \sup_{x \in E} |D^i \Phi(x)|_{\mathcal{L}^i(E)},$$

by using (2.4) we can conclude. □

Next, as

$$\|\varphi_v\|_{C_b^\theta(E)} \leq |v|_{E^*} \|\Phi\|_{C_b^\theta(E, E)},$$

by proceeding as we did in [6] by using interpolation, we obtain the following generalization of Proposition 2.3 to the vectorial case.

Proposition 3.2. *For any $\theta \in (0, 1)$ and $j = 2, 3$, there exists $c_{\theta, j} > 0$ such that for all $\Phi \in C_b^\theta(E, E)$ and all $t > 0$*

$$\|\widehat{P}_t \Phi\|_{C_b^j(E, E)} \leq c_{\theta, j} (t \wedge 1)^{-\frac{j-\theta}{2}} \|\Phi\|_{C_b^\theta(E, E)}. \quad (3.5)$$

Notice that, due to (3.4), by proceeding as in Lemma 2.2, we have that $D(\widehat{P}_t \Phi)(x) \in \mathcal{L}(H)$, for any $\Phi \in C_b(E, E)$, $x \in E$ and $t > 0$, and, thanks to (3.5), as in (2.9) we have that

$$|D(\widehat{P}_t \Phi)(x) - D(\widehat{P}_t \Phi)(y)|_{\mathcal{L}(H)} \leq c(t) (t \wedge 1)^{-\frac{2-\theta}{2}} \|\Phi\|_{C_b^\theta(E, E)} |x - y|_E, \quad x, y \in E, \quad (3.6)$$

for any $\Phi \in C_b^\theta(E, E)$.

Now, as in the case of P_t , we can define the infinitesimal generator of \widehat{P}_t , as the unique m -dissipative operator $\widehat{\mathcal{L}} : D(\widehat{\mathcal{L}}) \subset C_b(E, E) \rightarrow C_b(E, E)$ such that

$$R(\lambda, \widehat{\mathcal{L}}) = (\lambda - \widehat{\mathcal{L}})^{-1} = \widehat{F}(\lambda), \quad \lambda > 0,$$

where

$$\widehat{F}(\lambda) \Phi(x) = \int_0^\infty e^{-\lambda t} \widehat{P}_t \Phi(x) dt, \quad x \in E.$$

Due to (3.3), it is immediate to check that $\Phi \in D(\widehat{\mathcal{L}})$ if and only if $\langle \Phi, v \rangle_E \in D(\mathcal{L})$, for any $v \in E^*$, and

$$\langle \widehat{\mathcal{L}} \Phi, v \rangle_E = \mathcal{L} \varphi_v. \quad (3.7)$$

As for \mathcal{L} , we have that

$$\|R(\lambda, \widehat{\mathcal{L}}) \Phi\|_{C_b(E, E)} \leq \frac{1}{\lambda} \|\Phi\|_{C_b(E, E)}, \quad \Phi \in C_b(E, E).$$

As a consequence of (3.4),

$$\sup_{x \in E} |D(R(\lambda, \widehat{\mathcal{L}})\Phi)(x)|_{\mathcal{L}(E)} \leq \frac{c}{\sqrt{\lambda}} \|\Phi\|_{C_b(E, E)}, \quad (3.8)$$

and, from (3.6), as in (2.17), if $\Phi \in C_b^\theta(E, E)$ we get

$$|D(R(\lambda, \widehat{\mathcal{L}})\Phi)(x) - D(R(\lambda, \widehat{\mathcal{L}})\Phi)(y)|_{\mathcal{L}(H)} \leq \frac{c}{\lambda^{-\frac{\theta}{2}}} \|\Phi\|_{C_b^\theta(E, E)} |x - y|_E, \quad x, y \in E. \quad (3.9)$$

Moreover, as a consequence of Proposition 3.2, we have

Theorem 3.3. *Let $\Psi \in C_b^\theta(E, E)$, with $\theta \in (0, 1)$, and let $\Phi = R(\lambda, \widehat{\mathcal{L}})\Psi$, with $\lambda > 0$. Then we have $\Phi \in C_b^{2+\theta}(E, E)$ and there exists $c > 0$ (independent of Ψ) such that*

$$\|\Phi\|_{C_b^{2+\theta}(E, E)} \leq c \|\Psi\|_{C_b^\theta(E, E)}. \quad (3.10)$$

Finally, we would like to stress that, in view of (3.7), if Φ solves the equation

$$\lambda \Phi - \widehat{\mathcal{L}}\Phi = \Psi,$$

then for any $v \in E^*$ the function φ_v solves the equation

$$\lambda \varphi_v - \mathcal{L}\varphi_v = \psi_v.$$

Now, let $\Phi \in C_b(E_{\epsilon_1}, E_\epsilon)$, for some $\epsilon_1 \leq \epsilon_0$ and $\epsilon > 0$. According to (1.8), we have that $\Phi(X(t, x)) \in E_\epsilon$, for any $t > 0$ and $x \in E$. Therefore, as the mapping $x \in E \mapsto X(t, x) \in E_{\epsilon_1}$ is continuous, we have that

$$\Phi \in C_b(E_{\epsilon_1}, E_\epsilon) \implies \widehat{P}_t\Phi \in C_b(E, E_\epsilon), \quad t > 0.$$

In fact, we have the following smoothing property

Lemma 3.4. *If $\Phi \in C_b(E, E) \cap B_b(E_{\epsilon_1}, E_\epsilon)$, for some $\epsilon > 0$ and $\epsilon_1 \leq \epsilon_0$, then $\widehat{P}_t\Phi : E \rightarrow E_\epsilon$ is differentiable and*

$$\sup_{x \in E} |D(\widehat{P}_t\Phi)(x)|_{\mathcal{L}(E, E_\epsilon)} \leq c(t) (t \wedge 1)^{-\frac{1}{2}} \|\Phi\|_{B_b(E_{\epsilon_1}, E_\epsilon)}. \quad (3.11)$$

Therefore

$$\sup_{x \in E} |D(R(\lambda, \widehat{\mathcal{L}})\Phi)(x)|_{\mathcal{L}(E, E_\epsilon)} \leq c \lambda^{-\frac{1}{2}} \|\Phi\|_{B_b(E_{\epsilon_1}, E_\epsilon)}. \quad (3.12)$$

Proof. As $\Phi \in C_b(E, E)$, we have $\widehat{P}_t\Phi \in C_b^1(E, E)$ and for any $x, h \in E$

$$D(\widehat{P}_t\Phi)(x) \cdot h = \frac{1}{t} \mathbb{E} \Phi(X(t, x)) \int_0^t \langle D_x X(s, x) h, dw(s) \rangle_H.$$

Thanks to (1.18), as $\Phi \in B_b(E_{\epsilon_1}, E_\epsilon)$, we get

$$|D(\widehat{P}_t\Phi)(x) \cdot h|_{E_\epsilon} \leq \frac{c(t)}{t} \|\Phi\|_{B_b(E_{\epsilon_1}, E_\epsilon)} \sqrt{t} |h|_E.$$

This implies (3.11) and hence (3.12) .

□

Next, we introduce the vectorial semigroup in H , by defining

$$\hat{P}_t^H \Phi(x) = \mathbb{E} \Phi(X(t, x)), \quad x \in H, \quad t \geq 0,$$

for any $\Phi \in C_b(H, H)$, where $X(t, x)$ is the unique generalized solution of (1.1) in H . $\hat{\mathcal{L}}^H$ is the corresponding weak generator, defined as $\hat{\mathcal{L}}$.

By arguing as in the proof of Proposition 3.1, from (2.12) for any $0 \leq \alpha \leq \beta \leq 1$ we have

$$\|\hat{P}_t^H \Phi\|_{C_b^\beta(H, H)} \leq c(t) (t \wedge 1)^{-\frac{\beta-\alpha}{2}} \|\Phi\|_{C_b^\alpha(H, H)}, \quad t > 0. \quad (3.13)$$

and from Lemma 2.4 we have that \hat{P}_t^H maps $C_b^\alpha(H, H)$ into $C_b^{1+\beta}(H, H)$, for any $0 \leq \alpha \leq \beta < 1$, and

$$\|\hat{P}_t^H \Phi\|_{C_b^{1+\beta}(H, H)} \leq c(t) (t \wedge 1)^{-(\delta + \frac{\beta-\alpha}{2})} \|\Phi\|_{C_b^\alpha(H, H)}, \quad (3.14)$$

where δ is the constant defined in (2.14).

Finally, from Theorem 2.6, we get that if $\Psi \in C_b^\theta(H, H) \cap C_b^\alpha(E, E)$, with $\alpha > 0$ and $\theta > 1/2$, then the series $\sum_{i=1}^\infty D^2(R(\lambda, \hat{\mathcal{L}})\Psi)(x)(e_i, e_i)$ is convergent in H , uniformly with respect to $x \in B_R(E)$, and

$$\left| \sum_{i=1}^\infty D^2(R(\lambda, \hat{\mathcal{L}})\Psi)(x)(e_i, e_i) \right|_H \leq c \lambda^{\frac{\theta}{2} - \frac{1}{4}} (1 + |x|_E^{2m-1}) \|\Psi\|_{C_b^\theta(H, H)}. \quad (3.15)$$

4 Perturbations

We study now suitable perturbations of the Kolmogorov operator \mathcal{L} , obtained by adding a first order term. We distinguish the case the drift is regular and then in particular there is uniqueness for the corresponding stochastic equation, and the case the drift is only Hölder continuous.

4.1 Regular perturbations

We are here concerned with the operator

$$\hat{\mathcal{L}}\Phi + D\Phi \cdot B, \quad \Phi \in D(\hat{\mathcal{L}}), \quad (4.1)$$

where $B \in C_b^1(E, E)$.

We consider the stochastic differential equation

$$\begin{cases} dY(t) = [AY(t) + F(Y(t)) + B(Y(t))] dt + dw(t), \\ Y(0) = x \in E, \end{cases} \quad (4.2)$$

which we write in the following mild form

$$Y(t) = e^{tA}x + \int_0^t e^{(t-s)A}F(Y(s))ds + W_A(t) + \int_0^t e^{(t-s)A}B(Y(s))ds. \quad (4.3)$$

By reasoning as in [4, Proposition 6.2.2] for equation (1.5), equation (4.2) has a unique mild solution $Y(t, x) \in L^p(\Omega; C([0, T]; E))$, for any $p \geq 1$ and $T > 0$.

Next lemma shows that a stochastic non-linear variation of constants formula holds, which allows to write equation (4.2) in terms of the solution of equation (1.2) and of the associated first derivative equation. The proof, that we omit, follows from the same argument used in [2], adapted to this stochastic case.

Lemma 4.1. *Let $Y(t, x)$ and $X(t, x)$ be the solutions of equations (4.2) and (1.2), respectively. Then we have*

$$Y(t, x) = X(t, x) + \int_0^t U_{t,s}^{Y(s,x)} B(Y(s, x)) ds, \quad (4.4)$$

where $U_{t,s}^x h$ is the solution of the first derivative equation

$$\eta'(t) = A\eta(t) + F'(X(t, x))\eta(t), \quad \eta(s) = h,$$

for any $x \in E$ and $0 \leq s \leq t$ (see (1.12) and (1.13) and Lemma 1.2).

Now, we define the corresponding transition semigroup

$$\widehat{Q}_t \Phi(x) = \mathbb{E}[\Phi(Y(t, x))], \quad \Phi \in C_b(E, E), \quad (4.5)$$

whose infinitesimal generator $\widehat{\mathcal{N}}$ is defined in the same way we did before for the generator $\widehat{\mathcal{L}}$ of the semigroup \widehat{P}_t . This means that $\widehat{\mathcal{N}}$ is the m -dissipative operator in $C_b(E, E)$, whose domain $D(\widehat{\mathcal{N}})$ is characterized as the linear space of all functions $\Phi \in C_b(E, E)$ such that there exists the limit

$$\lim_{t \rightarrow 0} \frac{\widehat{Q}_t \Phi(x) - \Phi(x)}{t} = \widehat{\mathcal{N}} \Phi(x), \quad x \in E,$$

and

$$\sup_{t \in (0,1]} \sup_{x \in E} \frac{1}{t} \left| \widehat{Q}_t \Phi(x) - \Phi(x) \right|_E < \infty.$$

Notice that, as we are assuming $B \in C_b^2(E, E)$, the same arguments used for equation (1.2) and the semigroup \widehat{P}_t adapt to equation (4.2) and hence we have

$$\sup_{x \in E} |D(\widehat{Q}_t \Phi)(x)|_{\mathcal{L}(E)} \leq c(t \wedge 1)^{-\frac{1}{2}} \|\Phi\|_{C_b(E, E)}.$$

This implies that $D(\widehat{\mathcal{N}}) \subset C_b^1(E, E)$ and

$$\sup_{x \in E} |D((\lambda - \widehat{\mathcal{N}})^{-1} \Phi)(x)|_{\mathcal{L}(E)} \leq \frac{c}{\sqrt{\lambda}} \|\Phi\|_{C_b(E, E)}. \quad (4.6)$$

Proposition 4.2. *We have $D(\widehat{\mathcal{N}}) = D(\widehat{\mathcal{L}})$ and*

$$\widehat{\mathcal{N}} \Phi = \widehat{\mathcal{L}} \Phi + D\Phi \cdot B, \quad \Phi \in D(\widehat{\mathcal{L}}) = D(\widehat{\mathcal{N}}). \quad (4.7)$$

Proof. In view of Lemma 4.1 and of the fact that $D(\widehat{\mathcal{L}}) \subset C_b^1(E, E)$, we have for any $\Phi \in D(\widehat{\mathcal{L}})$

$$\begin{aligned}\Phi(X(t, x)) &= \Phi(Y(t, x)) - [\Phi(X(t, x) + R(t, x)) - \Phi(X(t, x))] \\ &= \Phi(Y(t, x)) - \int_0^1 D\Phi(X(t, x) + \theta R(t, x)) d\theta \cdot R(t, x),\end{aligned}$$

where

$$R(t, x) = \int_0^t U_{t,s}^{Y(s,x)} B(Y(s, x)) ds.$$

so that

$$\frac{\widehat{Q}_t \Phi(x) - \Phi(x)}{t} = \frac{\widehat{P}_t \Phi(x) - \Phi(x)}{t} + \mathbb{E} \int_0^1 D\Phi(X(t, x) + \theta R(t, x)) d\theta \cdot \frac{1}{t} R(t, x).$$

Now, for any $x \in E$ we have

$$\lim_{t \rightarrow 0} \mathbb{E} \int_0^1 D\Phi(X(t, x) + \theta R(t, x)) d\theta \cdot \frac{1}{t} R(t, x) = D\Phi(x) \cdot B,$$

and, due to (1.16), for $t \in (0, 1]$ we have

$$\left| \int_0^1 D\Phi(X(t, x) + \theta R(t, x)) d\theta \cdot \frac{1}{t} R(t, x) \right| \leq c \|B\|_{C_b(E, E)} \|\Phi\|_{C_b^1(E, E)}.$$

As we are assuming that $\Phi \in D(\widehat{\mathcal{L}})$, this allows us to conclude that

$$\lim_{t \rightarrow 0} \frac{\widehat{Q}_t \Phi(x) - \Phi(x)}{t} = (\mathcal{L}\Phi + D\Phi \cdot B)(x), \quad x \in E,$$

and hence $\Phi \in D(\widehat{\mathcal{N}})$ and (4.7) holds.

The inclusion $D(\widehat{\mathcal{N}}) \subset D(\widehat{\mathcal{L}})$ follows from an analogous argument, as $D(\widehat{\mathcal{N}}) \subset C_b^1(E, E)$. \square

4.2 Hölder perturbations

Now, we aim to study the elliptic equation

$$\lambda \Phi(x) - \widehat{\mathcal{L}}\Phi(x) - D\Phi(x) \cdot B(x) = G(x), \quad x \in E, \quad (4.8)$$

where $\lambda > 0$, $G \in C_b^\alpha(E, E)$ and $B \in C_b^\alpha(E; E)$, for some $\alpha \in (0, 1)$. We are going to show the following result.

Theorem 4.3. *Let $B \in C_b^\alpha(E; E)$ and $G \in C_b^\alpha(E, E)$, for some $\alpha \in (0, 1)$. Then, for any $\lambda > 0$ there exists a unique solution $\Phi \in D(\widehat{\mathcal{L}}) \cap C_b^{2+\alpha}(E, E)$ of equation (4.8). Moreover, for any $\epsilon \in [0, 2]$ there exists $c_\epsilon > 0$ (independent of λ and G) such that*

$$\|\Phi\|_{C_b^{2+\alpha-\epsilon}(E, E)} \leq c_\epsilon \left(\frac{1}{\lambda^{\epsilon/2}} + \frac{1}{\lambda} \right) \|G\|_{C_b^\alpha(E, E)}. \quad (4.9)$$

Proof. Step 1. Let $\Phi \in D(\widehat{\mathcal{N}}) \cap C_b^{2+\alpha}(E, E)$ be a solution of (4.8). Then we have

$$\|\Phi\|_{C_b(E, E)} \leq \frac{1}{\lambda} \|G\|_{C_b(E, E)}. \quad (4.10)$$

By an approximation result due to Valentine [18], we can choose a sequence $\{B_n\} \subset C_b^1(E, E)$ uniformly convergent to B . Then, thanks to Proposition 4.2 we can write equation (4.8) as

$$\lambda\Phi - \widehat{\mathcal{L}}\Phi - D\varphi \cdot B_n = G_n, \quad (4.11)$$

where

$$G_n(x) = G(x) + D\Phi|_E(x) \cdot (B(x) - B_n(x)), \quad x \in E. \quad (4.12)$$

Consider now the stochastic differential equation

$$\begin{cases} dX_n(t) = (AX_n(t) + F(X_n(t)) + B_n(X_n(t)))dt + dw(t), \\ X_n(0) = x \in H, \end{cases} \quad (4.13)$$

which has a unique solution $X_n(t, x)$. Then, if we introduce the transition semigroup

$$\widehat{Q}_t^n \Phi(x) = \mathbb{E} \Phi(X_n(t, x)), \quad \Phi \in C_b(E, E), \quad (4.14)$$

and the corresponding generator $\widehat{\mathcal{N}}_n$, we have

$$\lambda\Phi - \widehat{\mathcal{N}}_n \Phi = G_n.$$

Consequently,

$$\|\Phi\|_{C_b(E, E)} \leq \frac{1}{\lambda} \|G_n\|_{C_b(E, E)}.$$

Now the conclusion follows letting $n \rightarrow \infty$.

Step 2. There exists a constant $c > 0$ such that if $\Phi \in D(\widehat{\mathcal{L}}) \cap C_b^{2+\alpha}(E, E)$ is a solution of (4.8) then

$$\|\Phi\|_{C_b^{2+\alpha}(E, E)} \leq c \|G\|_{C_b^\alpha(E, E)}. \quad (4.15)$$

By (4.8) and Schauder's estimate (2.20), there exists $c > 0$ (independent of λ and f) such that

$$\|\Phi\|_{C_b^{2+\alpha}(E, E)} \leq c (\|G\|_{C_b^\alpha(E, E)} + \|B\|_{C_b^\alpha(E, E)} \|\Phi\|_{C_b^{1+\alpha}(E, E)}).$$

Now the conclusion follows from standard interpolatory estimates, as, by (4.10)

$$\begin{aligned} \|\Phi\|_{C_b^{2+\alpha}(E, E)} &\leq c \left(\|G\|_{C_b^\alpha(E, E)} + \|B\|_{C_b^\alpha(E, E)} \|\Phi\|_{C_b^{2+\alpha}(E, E)}^{\frac{1+\alpha}{2+\alpha}} \|\Phi\|_{C_b^{1+\alpha}(E, E)}^{\frac{1}{2+\alpha}} \right) \\ &\leq c' \|G\|_{C_b^\alpha(E, E)} + \frac{1}{2} \|\Phi\|_{C_b^{2+\alpha}(E, E)}. \end{aligned}$$

Step 3. For any $\epsilon \geq 0$, let us consider the equation

$$\lambda\Phi - \widehat{\mathcal{L}}\Phi - \epsilon D\Phi \cdot B = G. \quad (4.16)$$

Then, the set $\Lambda := \{\epsilon \in [0, 1] : (4.16) \text{ has a unique solution } \Phi \in D(\widehat{\mathcal{L}}) \cap C_b^{2+\alpha}(E)\}$ is open.

Assume $\epsilon_0 \in \Lambda$. We want to prove that for ϵ sufficiently close to ϵ_0 equation (4.16) has a unique solution. If we set

$$\lambda\Phi - \widehat{\mathcal{L}}\Phi - \epsilon_0 D\Phi \cdot B = \Psi, \quad (4.17)$$

equation (4.16) becomes

$$\Psi - T_{\lambda, \epsilon} \Psi = G, \quad (4.18)$$

where

$$T_{\lambda, \epsilon} \Psi(x) = (\epsilon - \epsilon_0) DR(\lambda, \mathcal{L}) \Psi \cdot B. \quad (4.19)$$

According to (2.16), we have

$$\begin{aligned} \|T_{\lambda, \epsilon} \Psi\|_{C_b(E, E)} &\leq |\epsilon_0 - \epsilon| \|B\|_{C_b(E, E)} \sup_{x \in E} \|D(R(\lambda, \widehat{\mathcal{L}})\Psi)(x)\|_{\mathcal{L}(E)} \\ &\leq \frac{|\epsilon_0 - \epsilon|}{\sqrt{\lambda}} \|B\|_{C_b(E, E)} \|\Psi\|_{C_b(E, E)}. \end{aligned}$$

and

$$\begin{aligned} [T_{\lambda, \epsilon} \Psi]_{C_b^\alpha(E, E)} &\leq |\epsilon_0 - \epsilon| \|B\|_{C_b(E, E)} [DR(\lambda, \widehat{\mathcal{L}})\Psi]_{C_b^\alpha(E, E)} \\ &+ |\epsilon_0 - \epsilon| \|B\|_{C_b^\alpha(E, E)} \|D(R(\lambda, \widehat{\mathcal{L}})\Psi)\|_{C_b(E, E)} \leq c \frac{|\epsilon_0 - \epsilon|}{\sqrt{\lambda}} \|B\|_{C_b^\alpha(E, E)} \|\Psi\|_{C_b^\alpha(E, E)}. \end{aligned}$$

Consequently

$$\|T_{\lambda, \epsilon} \Psi\|_{C_b^\alpha(E, E)} \leq \frac{2c |\epsilon_0 - \epsilon|}{\sqrt{\lambda}} \|B\|_{C_b^\alpha(E, E)} \|\Psi\|_{C_b^\alpha(E, E)} \quad (4.20)$$

so that $T_{\lambda, \epsilon}$ is a contraction on $C_b^\alpha(E)$ provided $|\epsilon_0 - \epsilon| < \sqrt{\lambda}/2c \|B\|_{C_b^\alpha(E, E)}$. By the contraction principle, this allows to conclude that there exists $\Psi \in C_b^\alpha(E, E)$ solving (4.18) and, as $\epsilon_0 \in \Lambda$, this implies that there exists a unique solution Φ for equation (4.8), which belongs to $C_b^{2+\alpha}(E, E)$.

Step 4. Conclusion.

We use the continuity method. The set Λ introduced above is non empty, as $0 \in \Lambda$. Moreover, due to the previous step, it is open. Therefore, if we show that is closed, we have $\Lambda = [0, 1]$ and the conclusion follows. Let $\epsilon_n \rightarrow \bar{\epsilon}$ with $(\epsilon_n) \subset \Lambda$. We have

$$\lambda(\Phi_{\epsilon_n} - \Phi_{\epsilon_m}) - \widehat{\mathcal{L}}(\Phi_{\epsilon_n} - \Phi_{\epsilon_m}) - \epsilon_n D(\Phi_{\epsilon_n} - \Phi_{\epsilon_m}) \cdot B = (\epsilon_n - \epsilon_m) D\Phi_{\epsilon_m} \cdot B.$$

From the Schauder estimate (2.20) and (4.15), we get

$$\begin{aligned} \|\Phi_{\epsilon_n} - \Phi_{\epsilon_m}\|_{C_b^{2+\alpha}(E, E)} &\leq c |\epsilon_n - \epsilon_m| \|B\|_{C_b^\alpha(E, E)} \|\Phi_{\epsilon_m}\|_{C_b^{1+\alpha}(E, E)} \\ &\leq c |\epsilon_n - \epsilon_m| \|B\|_{C_b^\alpha(E, E)} \|G\|_{C_b^\alpha(E, E)}, \end{aligned}$$

and then we conclude that $\{\Phi_{\epsilon_n}\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $C_b^{2+\alpha}(E, E)$. This implies that the sequence $\{\Phi_{\epsilon_n}\}_{n \in \mathbb{N}}$ converges to some $\bar{\Phi} \in D(\hat{\mathcal{L}}) \cap C_b^{2+\alpha}(E, E)$ and such $\bar{\Phi}$ is the unique solution of equation (4.16), for $\bar{\epsilon}$.

Step 5. Proof of estimate (4.9).

Due to (3.5), we have

$$\|R(\lambda, \hat{\mathcal{L}})\Phi\|_{C_b^{2+\alpha-\epsilon}(E, E)} \leq c \int_0^\infty e^{-\lambda t} (t \wedge 1)^{-1+\frac{\epsilon}{2}} dt \|G\|_{C_b^\alpha(E, E)},$$

so that (4.9) follows immediately. \square

In fact, the solution Φ of equation (4.8) satisfies the following properties.

Lemma 4.4. *Assume that $B, G \in C_b^\alpha(E, E)$, for some $\alpha > 0$. Then, if Φ is the solution of equation (4.8), if λ is large enough then $D\Phi(x) \in \mathcal{L}(H)$, for any $x \in E$, and*

$$|D\Phi(x) - D\Phi(y)|_{\mathcal{L}(H)} \leq c|x - y|_E, \quad x, y \in E. \quad (4.21)$$

Moreover, if we also assume that $G \in B_b(E_{\epsilon_1}, E_\epsilon)$, for some $\epsilon > 0$ and $\epsilon_1 \leq \epsilon_0$, then

$$|\Phi(x) - \Phi(y)|_{E_\epsilon} \leq c\lambda^{-\frac{1}{2}}\|G\|_{C_b(E_\epsilon, E_\epsilon)}|x - y|_E, \quad x, y \in E. \quad (4.22)$$

Proof. By proceeding as in Step 3 in the proof of Theorem 4.3, for λ large enough the mapping

$$T_\lambda : C_b^\alpha(E, E) \rightarrow C_b^\alpha(E, E), \quad \Psi \mapsto T_\lambda \Psi = D(R(\lambda, \hat{\mathcal{L}})\Psi) \cdot B,$$

is a contraction. Therefore, as $\Phi = R(\lambda, \hat{\mathcal{L}})(I - T_\lambda)^{-1}G$, due to (3.9) we have that $D\Phi(x) \in \mathcal{L}(H)$, for any $x \in E$, and (4.21) holds.

In view of Lemma 3.4 and (3.12), we have that for any $\Psi \in C_b(E, E) \cap B_b(E_{\epsilon_1}, E_\epsilon)$ the mapping $x \in E \mapsto D(R(\lambda, \hat{\mathcal{L}})\Psi)(x) \cdot B(x) \in E_\epsilon$ is well defined and continuous, and

$$\sup_{x \in E} |D(R(\lambda, \hat{\mathcal{L}})\Psi)(x) \cdot B(x)|_{E_\epsilon} \leq c\lambda^{-\frac{1}{2}}\|\Psi\|_{B_b(E_{\epsilon_1}, E_\epsilon)}.$$

This implies that if λ is large enough

$$T_\lambda : C_b(E, E) \cap B_b(E_{\epsilon_1}, E_\epsilon) \rightarrow C_b(E, E) \cap B_b(E_{\epsilon_1}, E_\epsilon),$$

is a contraction. Therefore, as $\Phi = R(\lambda, \hat{\mathcal{L}})(I - T_\lambda)^{-1}G$, due to (3.12) we have that Φ is continuous from E into E_ϵ .

Now, for any $x, y \in E$, we have

$$\Phi(x) - \Phi(y) = \int_0^1 D\Phi(\theta x + (1 - \theta)y) \cdot (x - y) d\theta,$$

and then, as

$$D\Phi(\theta x + (1 - \theta)y) \cdot (x - y) = D(R(\lambda, \hat{\mathcal{L}})(I - T_\lambda)^{-1}G)(\theta x + (1 - \theta)y) \cdot (x - y),$$

according to (3.12) we conclude

$$|\Phi(x) - \Phi(y)|_{E_\epsilon} \leq c\lambda^{-\frac{1}{2}}\|(I - T_\lambda)^{-1}G\|_{B_b(E_{\epsilon_1}, E_\epsilon)}|x - y|_E.$$

\square

Finally, we show that under stronger assumptions on B and G , the solution Φ of equation (4.8) has some further properties.

Theorem 4.5. *Assume that $B, G \in C_b^\alpha(E, E) \cap C_b^\theta(H, H)$, for some $\theta \in [0, 1)$, and take λ sufficiently large. Then,*

1. *we have $\Phi \in C_b^{1+\theta}(H, H)$;*
2. *if $\theta > 1/2$, the series $\sum_{i=1}^\infty D^2\Phi(x)(e_i, e_i)$ and $\sum_{i=1}^\infty D^2\Phi(J_n x)(J_n e_i, J_n e_i)$ are convergent in H , uniformly with respect to $n \in \mathbb{N}$ and $x \in B_R(E)$, for any $R > 0$. In particular, for any $x \in E$*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^\infty D^2\Phi(J_n x)(J_n e_i, J_n e_i) = \sum_{i=1}^\infty D^2\Phi(x)(e_i, e_i) \quad \text{in } E. \quad (4.23)$$

Proof. Proof of 1. According to (3.14) we have that \hat{P}_t maps $C_b^\theta(H, H)$ into $C_b^{1+\theta}(H, H)$ with

$$\|\hat{P}_t \Psi\|_{C_b^{1+\theta}(H, H)} \leq c(t)(t \wedge 1)^{-\delta} \|\Psi\|_{C_b^\theta(H, H)},$$

where δ is the constant, strictly less than 1, defined in (2.14). This implies that

$$R(\lambda, \hat{\mathcal{L}}) : C_b^\theta(H, H) \rightarrow C_b^{1+\theta}(H, H), \quad (4.24)$$

and

$$\sup_{x \in H} |D(R(\lambda, \hat{\mathcal{L}})\Psi)(x)h|_H \leq \frac{c}{\lambda^{\delta-1}} |h|_H \|\Psi\|_{C_b(H, H)},$$

Hence, as we are assuming $B \in C_b^\theta(H, H)$, if we pick λ large enough, we have that the mapping

$$T_\lambda : C_b^\theta(H, H) \rightarrow C_b^\theta(H, H), \quad \Psi \mapsto T_\lambda \Psi = D(R(\lambda, \hat{\mathcal{L}})\Psi) \cdot B,$$

is a contraction. Now, as $\Phi = R(\lambda, \hat{\mathcal{L}})(I - T_\lambda)^{-1}G$ and $G \in C_b^\theta(H, H)$, thanks to (4.24), we conclude that $\Phi \in C_b^{1+\theta}(H, H)$.

Proof of 2. Due to the previous step, we have $D\Phi \cdot B + G \in C_b^\theta(H, H)$. Then, as we have

$$\Phi = R(\lambda, \hat{\mathcal{L}})[D\Phi \cdot B + G],$$

and we are assuming $\theta > 1/2$, we can conclude from (3.15) and from Remark 2.7. □

5 Pathwise uniqueness

We want to prove that pathwise uniqueness holds in the class of mild solutions of the equation

$$\begin{cases} dY(t) = [AY(t) + F(Y(t)) + B(Y(t))]dt + dw(t), \\ Y(0) = x, \end{cases} \quad (5.1)$$

where A , F and W are as in section 1 and B satisfies the following condition.

Hypothesis 2. *There exist $\alpha, \epsilon > 0$ and $\epsilon_1 \leq \epsilon_0$ such that*

$$B \in C_b^\alpha(E, E) \cap B_b(E_{\epsilon_1}, E_\epsilon).$$

Remark 5.1. We have already seen that the mappings B described in Subsection 0.1 are both in $C_b^\alpha(E, E)$. Moreover, they belong to $B_b(E_{\epsilon_1}, E_\epsilon)$, for suitable positive constants as in Hypothesis 2.

Let

$$B(x)(\xi) = b(x(\xi_0))g(\xi), \quad \xi \in [0, 1],$$

for some $g \in E$, $\xi_0 \in [0, 1]$ and $b \in C_b^\alpha(\mathbb{R}, \mathbb{R})$. If we assume that $g \in E_\epsilon$, then B maps E_ϵ into E_ϵ . Actually, for any $\xi_1, \xi_2 \in [0, 1]$ we have

$$B(x)(\xi_1) - B(x)(\xi_2) = b(x(\xi_0))(g(\xi_1) - g(\xi_2)),$$

so that

$$[B(x)]_{E_\epsilon} \leq \|b\|_\infty [g]_{E_\epsilon}.$$

Now, let

$$B(x)(\xi) = b\left(\max_{s \in [0, \xi]} x(s)\right), \quad \xi \in [0, 1],$$

for some $b \in C_b^\alpha(\mathbb{R}, \mathbb{R})$. Then, B maps E_ϵ into $E_{\epsilon\alpha}$. Actually, for any $\xi_1, \xi_2 \in [0, 1]$, with $\xi_1 > \xi_2$, we have

$$\begin{aligned} |B(x)(\xi_1) - B(x)(\xi_2)| &\leq [b]_{C^\alpha(\mathbb{R})} \left| \max_{s \leq \xi_1} x(s) - \max_{s \leq \xi_2} x(s) \right|^\alpha \\ &= [b]_{C^\alpha(\mathbb{R})} (x(\bar{\xi}_1) - x(\bar{\xi}_2))^\alpha, \end{aligned}$$

for some $\bar{\xi}_i \leq \xi_i$, $i = 1, 2$. If $\bar{\xi}_1 \leq \xi_2$, then $x(\bar{\xi}_1) = x(\bar{\xi}_2)$ and we are done. Thus, assume $\xi_2 \leq \bar{\xi}_1 \leq \xi_1$. We have

$$0 \leq x(\bar{\xi}_1) - x(\bar{\xi}_2) \leq x(\bar{\xi}_1) - x(\xi_2) \leq |x(\bar{\xi}_1) - x(\xi_2)| \leq |\bar{\xi}_1 - \xi_2|^\epsilon \leq |\xi_1 - \xi_2|^\epsilon.$$

As in [8], the main idea here is to represent the *bad term* $B(Y(t))$ in terms of nicer objects, by using Itô's formula.

To this purpose, we show how we can point-wise approximate the mapping B by nicer mappings B_m .

Lemma 5.2. *Under Hypothesis 2, there exists a sequence $\{B_m\}_{m \in \mathbb{N}} \subset C_b^\alpha(E, E) \cap C_b^\infty(H, E)$ such that*

$$\begin{cases} \lim_{m \rightarrow \infty} |B_m(x) - B(x)|_E = 0, & x \in E, \\ \sup_{m \in \mathbb{N}} \|B_m\|_{C_b^\alpha(E, E)} < \infty. \end{cases}$$

Proof. For any $m \in \mathbb{N}$ we define

$$T_m \xi = \sum_{k=1}^m \xi_k e_k, \quad \xi \in \mathbb{R}^m, \quad Q_m x = (x_1, \dots, x_m), \quad P_m x = \sum_{k=1}^m x_k e_k, \quad x \in H,$$

where $x_k = \langle x, e_k \rangle_H$. If we define

$$\hat{P}_m x = \frac{1}{m} \sum_{k=1}^m P_k x, \quad x \in H,$$

then Fejér's Theorem states that $\hat{P}_m x$ converges to x in E , as $m \uparrow \infty$, when $x \in E$. In particular, as a consequence of the uniform boundedness theorem,

$$\sup_{m \in \mathbb{N}} \|\hat{P}_m\|_{\mathcal{L}(E)} < \infty. \quad (5.2)$$

Now, as for any $x \in H$ we have $\hat{P}_m x \in E$, we can define

$$B_m(x) = \int_{\mathbb{R}^m} B(\hat{P}_m(x - T_m \xi)) \rho_m(\xi) d\xi, \quad x \in H,$$

where $\rho_m \in C_c^\infty(\mathbb{R}^m)$ is a probability density with support in $\{\xi \in \mathbb{R}^m, |\xi|_{\mathbb{R}^m} \leq 1/m^2\}$. We have clearly that $B_m : H \rightarrow E$ and due to (5.2) for any $x, y \in E$

$$\begin{aligned} |B_m(x) - B_m(y)|_E &\leq \int_{\mathbb{R}^m} |B(\hat{P}_m(x - T_m \xi)) - B(\hat{P}_m(y - T_m \xi))|_E \rho_m(\xi) d\xi \\ &\leq [B]_{C^\alpha(E, E)} |\hat{P}_m x - \hat{P}_m y|_E^\alpha \int_{\mathbb{R}^m} \rho_m(\xi) d\xi \leq c |x - y|_E^\alpha. \end{aligned}$$

This implies that $\{B_m\}_{m \in \mathbb{N}}$ is a bounded sequence in $C^\alpha(E, E)$.

Moreover, as $\hat{P}_{m_1} P_{m_2} = \hat{P}_{m_1}$, for any $m_1 \leq m_2$, with a change of variable we have

$$B_m(x) = \int_{\mathbb{R}^m} B(\hat{P}_m(x - T_m \xi)) \rho_m(\xi) d\xi = \int_{\mathbb{R}^m} B(\hat{P}_m T_m \eta) \rho_m(\eta + Q_m x) d\eta,$$

and, as ρ_m is in $C_c^\infty(\mathbb{R}^m)$, this implies that $B_m \in C_b^\infty(H, E)$.

Finally, for any $x \in E$ we have

$$|B_m(x) - B(x)|_E \leq c_\alpha [B]_{C^\alpha(E, E)} \int_{\mathbb{R}^m} \left(|\hat{P}_m x - x|_E^\alpha + |\hat{P}_m T_m \xi|_E^\alpha \right) \rho_m(\xi) d\xi,$$

and then, as for any $|\xi|_{\mathbb{R}^m} \leq 1/m^2$ we have

$$|\hat{P}_m T_m \xi|_E \leq \frac{1}{m} \sum_{k \leq m} \sum_{i \leq k} |\xi_i| \leq \frac{m(m+1)}{2m^3} \rightarrow 0, \quad \text{as } m \rightarrow \infty,$$

recalling that $|\hat{P}_m x - x|_E \rightarrow 0$, we conclude that $B_m(x)$ converges to $B(x)$ in E , for any $x \in E$. \square

Now we define

$$Y_n(t, x) = J_n Y(t, x),$$

where $J_n = nR(n, A)$, we have

$$\begin{cases} dY_n(t) = [AY_n(t) + J_n F(Y(t)) + J_n B(Y(t))] dt + J_n dw(t), \\ Y_n(0) = J_n x. \end{cases} \quad (5.3)$$

Notice that, if $Y(t, x)$ is a mild solution of equation (5.1), we have that $Y_n(t, x)$ is a strong solution of equation (5.3), that is

$$Y_n(t, x) = J_n x + \int_0^t [AY_n(s, x) + J_n F(Y(s, x)) + J_n B(Y(s, x))] ds + J_n W(t). \quad (5.4)$$

Moreover,

$$\begin{cases} |Y_n(t, x)|_E \leq |Y(t, x)|_E, \\ \lim_{n \rightarrow \infty} |Y_n(t, x) - Y(t, x)|_E = 0, \end{cases} \quad (5.5)$$

for any $t \geq 0$ and $x \in E$, \mathbb{P} -a.s.

Now, for each $\lambda > 0$ we consider the elliptic equation

$$\lambda \Phi_m - \widehat{\mathcal{L}} \Phi_m - D\Phi_m \cdot B_m = B_m, \quad (5.6)$$

where B_m is the mapping introduced in Lemma 5.2. Later on we will choose $\lambda > 0$ large enough. We denote by Φ_m its unique solution. According to what we have seen in Theorem 4.3, we have that $\Phi_m \in C_b^{2+\alpha}(E, E)$ and as the sequence $\{B_m\}_{m \in \mathbb{N}}$ is equi-bounded in $C_b^\alpha(E, E)$, we have

$$\sup_{m \in \mathbb{N}} \|\Phi_m\|_{C_b^{2+\alpha}(E, E)} < \infty. \quad (5.7)$$

Lemma 5.3. *If λ is large enough, we have*

$$\lim_{m \rightarrow \infty} |\Phi_m(x) - \Phi(x)|_E = 0, \quad x \in E, \quad (5.8)$$

and

$$\lim_{m \rightarrow \infty} |D\Phi(x) - D\Phi_m(x)|_{\mathcal{L}(H, E)} = 0, \quad x \in E. \quad (5.9)$$

Proof. We have

$$\Phi - \Phi_m = R(\lambda, \widehat{\mathcal{L}})(\Psi - \Psi_m),$$

where

$$\Psi = (I - T_\lambda)^{-1} B, \quad \Psi_m = (I - T_{\lambda, m})^{-1} B_m,$$

and

$$T_\lambda \Psi(x) = D(R(\lambda, \widehat{\mathcal{L}})\Psi)(x) \cdot B(x), \quad T_{\lambda, m} \Psi(x) = D(R(\lambda, \widehat{\mathcal{L}})\Psi)(x) \cdot B_m(x), \quad x \in E.$$

Therefore, if we show that the sequence $\{\Psi_m\}_{m \in \mathbb{N}}$ is bounded in $C_b(E, E)$ and

$$\lim_{m \rightarrow \infty} |\Psi(x) - \Psi_m(x)|_E = 0, \quad (5.10)$$

in view of what we have seen in Section 3, it is immediate to check that

$$\lim_{m \rightarrow \infty} |R(\lambda, \widehat{\mathcal{L}})(\Psi - \Psi_m)(x)|_E = 0, \quad x \in E, \quad (5.11)$$

and

$$\lim_{m \rightarrow \infty} |D(R(\lambda, \widehat{\mathcal{L}})(\Psi - \Psi_m))(x)|_{\mathcal{L}(H, E)} = 0, \quad x \in E, \quad (5.12)$$

and (5.8) and (5.9) follow.

We have

$$\begin{aligned} (\Psi - \Psi_m) - T_\lambda(\Psi - \Psi_m) &= [\Psi - T_\lambda \Psi] - [\Psi_m - T_{\lambda, m} \Psi_m] + D(R(\lambda, \widehat{\mathcal{L}})\Psi_m) \cdot (B - B_m) \\ &= \left[I + D(R(\lambda, \widehat{\mathcal{L}})\Psi_m) \right] \cdot (B_m - B). \end{aligned}$$

If $\lambda > 0$ is large enough, the mapping $T_\lambda : C_b(E, E) \rightarrow C_b(E, E)$ is a contraction and then

$$\Psi - \Psi_m = (I - T_\lambda)^{-1} \left[I + D(R(\lambda, \widehat{\mathcal{L}})\Psi_m) \right] \cdot (B_m - B).$$

Now, due to (3.8), for any $x \in E$ we have

$$|D(R(\lambda, \widehat{\mathcal{L}})\Psi_m) \cdot (B_m - B)(x)|_E \leq \frac{c}{\sqrt{\lambda}} \|\Psi_m\|_{C_b(E, E)} |B_m(x) - B(x)|_E,$$

so that

$$\lim_{m \rightarrow \infty} \left| \left[I + D(R(\lambda, \widehat{\mathcal{L}})\Psi_m) \right] \cdot (B_m - B)(x) \right|_E = 0, \quad x \in E,$$

and

$$\sup_{m \in \mathbb{N}} \left\| \left[I + D(R(\lambda, \widehat{\mathcal{L}})\Psi_m) \right] \cdot (B_m - B) \right\|_{C_b(E, E)} < \infty.$$

According to (5.12), this implies

$$\lim_{m \rightarrow \infty} |T_\lambda \left[I + D(R(\lambda, \widehat{\mathcal{L}})\Psi_m) \right] \cdot (B_m - B)(x)|_E = 0, \quad x \in E.$$

Therefore, as

$$\begin{aligned} &(I - T_\lambda)^{-1} \left[I + D(R(\lambda, \widehat{\mathcal{L}})\Psi_m) \right] \cdot (B_m - B) \\ &= \sum_{k=0}^{\infty} T_\lambda^k \left[I + D(R(\lambda, \widehat{\mathcal{L}})\Psi_m) \right] \cdot (B_m - B), \end{aligned}$$

and T_λ is a contraction, we conclude that

$$\lim_{m \rightarrow \infty} |(I - T_\lambda)^{-1} \left[I + D(R(\lambda, \widehat{\mathcal{L}})\Psi_m) \right] \cdot (B_m - B)(x)|_E = 0, \quad x \in E,$$

so that (5.10) follows. □

As Φ_m belongs to $C_b^2(E, E)$, and $X_n(t, x)$ solves equation (5.4), we can use the generalization of the Ito formula in the space of continuous functions, proved in [7, Appendix A], and we have

$$d\Phi_m(Y_n(t, x)) = \left[\frac{1}{2} \sum_{i=1}^{\infty} D^2\Phi_m(Y_n(t, x))(e_i, e_i) + D\Phi_m \cdot B_m(Y_n(t, x)) \right] dt \\ + D\Phi_m(Y_n(t, x)) \cdot J_n dw(t) + R_{n,m}(t) dt,$$

where

$$R_{n,m}(t) = D\Phi_m(Y_n(t, x)) \cdot [J_n F(Y(t, x)) - F(Y_n(t, x))] \\ + D\Phi_m(Y_n(t, x)) \cdot [J_n B(Y(t, x)) - B_m(Y_n(t, x))] \\ + \frac{1}{2} \sum_{i=1}^{\infty} D^2\Phi_m(Y_n(t, x))(J_n e_i, J_n e_i) - \frac{1}{2} \sum_{i=1}^{\infty} D^2\Phi_m(Y_n(t, x))(e_i, e_i).$$

Therefore, since Φ_m solves equation (5.6), we have

$$d\Phi_m(Y_n(t, x)) = \lambda \Phi_m(Y_n(t, x)) dt - B_m(Y_n(t, x)) dt \\ + D\Phi_m(Y_n(t, x)) \cdot J_n dw(t) + R_{n,m}(t) dt,$$

and then, for any $0 \leq s \leq t$ we have

$$d e^{(t-s)A} \Phi_m(Y_n(s, x)) = (\lambda - A) e^{(t-s)A} \Phi_m(Y_n(s, x)) ds - e^{(t-s)A} B_m(Y_n(s, x)) ds \\ + e^{(t-s)A} D\Phi_m(Y_n(s, x)) \cdot J_n dw(s) + e^{(t-s)A} R_{n,m}(s) ds.$$

This yields

$$\int_0^t e^{(t-s)A} B_m(Y_n(s, x)) ds = e^{tA} \Phi_m(J_n x) - \Phi_m(Y_n(t, x)) \\ + \int_0^t (\lambda - A) e^{(t-s)A} \Phi_m(Y_n(s, x)) ds + \int_0^t e^{(t-s)A} D\Phi_m(Y_n(s, x)) \cdot J_n dw(s) \quad (5.13) \\ + \int_0^t e^{(t-s)A} R_{n,m}(s) ds.$$

Lemma 5.4. *Under Hypotheses 1 and 2, if $Y(t, x)$ is a mild solution of (5.1), we have*

$$\int_0^t e^{(t-s)A} B(Y(s, x)) ds = e^{tA} \Phi(x) - \Phi(Y(t, x)) \quad (5.14) \\ + \int_0^t (\lambda - A) e^{(t-s)A} \Phi(Y(s, x)) ds + \int_0^t e^{(t-s)A} D\Phi(Y(s, x)) \cdot dw(s),$$

where Φ is the unique solution of the elliptic equation

$$\lambda \Phi - \widehat{\mathcal{L}}\Phi - D\Phi \cdot B = B. \quad (5.15)$$

Proof. We first take the limit in (5.13) as n goes to infinity and then the limit as m goes to infinity.

Step 1: Limit as n goes to infinity

Due to a maximal regularity result, we have

$$\begin{aligned} & \int_0^T \left| \int_0^t (\lambda - A) e^{(t-s)A} \Phi_m(Y_n(s, x)) ds - \int_0^t (\lambda - A) e^{(t-s)A} \Phi_m(Y(s, x)) ds \right|_H^2 dt \\ & \leq c_\lambda(T) \int_0^T |\Phi_m(Y_n(s, x)) - \Phi_m(Y(s, x))|_H^2 ds, \end{aligned}$$

for some constant $c_\lambda(T)$, independent of Φ_m and $Y(t, x)$. Then, according to (5.5) and (5.7), we conclude

$$\lim_{n \rightarrow \infty} \int_0^\cdot (\lambda - A) e^{(\cdot-s)A} \Phi_m(Y_n(s, x)) ds = \int_0^\cdot (\lambda - A) e^{(\cdot-s)A} \Phi_m(Y(s, x)) ds. \quad (5.16)$$

\mathbb{P} -a.s. in $L^2(0, T; H)$.

Next, as $\Phi_m \in C_b^{1+\theta}(H, H)$, due to Theorem 4.5 we have

$$\begin{aligned} & \mathbb{E} \left| \int_0^t e^{(t-s)A} D\Phi_m(Y_n(s, x)) \cdot J_n dw(s) - \int_0^t e^{(t-s)A} D\Phi_m(Y(s, x)) \cdot J_n dw(s) \right|_H^2 \\ & = \mathbb{E} \int_0^t \sum_{i=1}^\infty \left| e^{(t-s)A} [D\Phi_m(Y_n(s, x)) - D\Phi_m(Y(s, x))] J_n e_i \right|_H^2 ds \\ & \leq \mathbb{E} \int_0^t \sum_{i=1}^\infty \sum_{j=1}^\infty \left| \left\langle [D\Phi_m(Y_n(s, x)) - D\Phi_m(Y(s, x))] e_i, e^{(t-s)A} e_j \right\rangle_H \right|^2 \\ & = \mathbb{E} \int_0^t \sum_{j=1}^\infty e^{-2(t-s)\alpha_j} \sum_{i=1}^\infty \left| \langle e_i, [D\Phi_m(Y_n(s, x)) - D\Phi_m(Y(s, x))]^* e_j \rangle_H \right|^2 \\ & \leq c(t) \int_0^t (t-s)^{-\frac{1}{2}} \mathbb{E} |Y_n(s, x) - Y(s, x)|_H^2 ds. \end{aligned}$$

As clearly

$$\mathbb{E} \left| \int_0^t e^{(t-s)A} D\Phi_m(Y(s, x)) \cdot J_n dw(s) - \int_0^t e^{(t-s)A} D\Phi_m(Y(s, x)) \cdot dw(s) \right|_H^2 = 0,$$

due to (5.5) this implies that for any $t \in [0, T]$

$$\lim_{n \rightarrow \infty} \int_0^t e^{(t-s)A} D\Phi_m(Y_n(s, x)) \cdot J_n dw(s) = \int_0^t e^{(t-s)A} D\Phi_m(Y(s, x)) \cdot dw(s), \quad (5.17)$$

in $L^2(\Omega; H)$.

Finally, as according to (4.23) we immediately have

$$\lim_{n \rightarrow \infty} R_{n,m}(t) = D\Phi_m(Y(t, x)) \cdot [B(Y(t, x)) - B_m(Y(t, x))],$$

from the dominated convergence theorem we have that for any $t \in [0, T]$

$$\lim_{n \rightarrow \infty} \int_0^t e^{(t-s)A} R_{n,m}(s) ds = \int_0^t e^{(t-s)A} D\Phi_m(Y(s, x)) \cdot [B(Y(s, x)) - B_m(Y(s, x))] ds, \quad (5.18)$$

\mathbb{P} -a.s. in H .

Therefore, collecting together (5.16), (5.17) and (5.18), from (5.7) and (5.13) we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^t e^{(t-s)A} B_m(Y_n(s, x)) ds &= e^{tA} \Phi_m(x) - \Phi_m(Y(t, x)) \\ &+ \int_0^t (\lambda - A) e^{(t-s)A} \Phi_m(Y(s, x)) ds + \int_0^t e^{(t-s)A} D\Phi_m(Y(s, x)) \cdot dw(s) \\ &+ \int_0^t e^{(t-s)A} D\Phi_m(Y(s, x)) \cdot [B(Y(s, x)) - B_m(Y(s, x))] ds, \end{aligned}$$

in $L^2(0, T; L^2(\Omega; H))$.

Step 2: Limit as m goes to infinity

By using arguments analogous to those used in the previous step, from Lemma 5.3 we have

$$\begin{aligned} \int_0^t e^{(t-s)A} B(Y(s, x)) ds &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \int_0^t e^{(t-s)A} B_m(Y_n(s, x)) ds = e^{tA} \Phi(x) - \Phi(Y(t, x)) \\ &+ \int_0^t (\lambda - A) e^{(t-s)A} \Phi(Y(s, x)) ds + \int_0^t e^{(t-s)A} D\Phi(Y(s, x)) \cdot dw(s), \end{aligned}$$

in $L^2(0, T; L^2(\Omega; H))$. This allows to conclude that (5.14) holds. □

The previous lemma has provided a nice representation of the *bad term*

$$\int_0^t e^{(t-s)A} B(Y(s, x)) ds,$$

for any mild solution $Y(t, x)$ of equation (5.1) and this allows to conclude that pathwise uniqueness holds.

Theorem 5.5. *Under Hypotheses 1 and 2, pathwise uniqueness holds for equation (5.1).*

Proof. Let $Y_1(t, x)$ and $Y_2(t, x)$ be two mild solutions of equation (5.1) in $L^2(\Omega; C([0, T]; E))$. Thanks to Lemma 5.4 we have

$$\begin{aligned} Y_1(t, x) - Y_2(t, x) &= \Phi(Y_2(t, x)) - \Phi(Y_1(t, x)) \\ &+ \int_0^t e^{(t-s)A} [F(Y_1(s, x)) - F(Y_2(s, x))] ds \\ &+ \int_0^t (\lambda - A) e^{(t-s)A} [\Phi(Y_1(s, x)) - \Phi(Y_2(s, x))] ds \\ &+ \int_0^t e^{(t-s)A} [D\Phi(Y_1(s, x)) - D\Phi(Y_2(s, x))] \cdot dw(s) =: \sum_{i=1}^4 I_i(t), \end{aligned}$$

where Φ is the unique solution of the elliptic equation (5.15).

Now, for any $R > 0$ we denote by τ_R the stopping time

$$\tau_R := \inf \{ t \geq 0, |Y_1(t, x)|_E \vee |Y_2(t, x)|_E \geq R \},$$

and we define

$$\tau := \lim_{R \rightarrow \infty} \tau_R.$$

Clearly, we have $\mathbb{P}(\tau = T) = 1$.

Step 1. According to (4.9), with $\epsilon = 1 + \alpha$, we have

$$|\Phi(x) - \Phi(y)|_E \leq c(\lambda) |x - y|_E,$$

for some function $c(\lambda) \downarrow 0$, as $\lambda \uparrow \infty$. Therefore, for any $p \geq 1$ there exists $\lambda_p > 0$ such that

$$|I_1(t)|_E^p \leq \frac{1}{2} |Y_1(t, x) - Y_2(t, x)|_E^p, \quad t \geq 0, \quad \lambda \geq \lambda_p. \quad (5.19)$$

Step 2. As F is locally Lipschitz-continuous in E , for any $R > 0$ we have

$$\begin{aligned} |I_2(t \wedge \tau_R)|_E &\leq c \int_0^{t \wedge \tau_R} (1 + |Y_1(s, x)|_E^{2m} + |Y_2(s, x)|_E^{2m}) |Y_1(s, x) - Y_2(s, x)|_E ds \\ &\leq c(1 + R^{2m}) \int_0^t |Y_1(s \wedge \tau_R, x) - Y_2(s \wedge \tau_R, x)|_E ds, \end{aligned}$$

so that, for any $p \geq 1$

$$|I_2(t \wedge \tau_R)|_E^p \leq c_{R,p}(t) \int_0^t |Y_1(s \wedge \tau_R, x) - Y_2(s \wedge \tau_R, x)|_E^p ds. \quad (5.20)$$

Step 3. By a factorization argument, for any $R > 0$ and $\beta \in (0, 1)$ we have

$$I_3(t \wedge \tau_R) = c_\beta \int_0^{t \wedge \tau_R} (t \wedge \tau_R - s)^{\beta-1} (\lambda - A) e^{(t \wedge \tau_R - s)A} v_\beta(s) ds,$$

where

$$v_\beta(s) = \int_0^s (s - \sigma)^{-\beta} (\lambda - A) e^{(s-\sigma)A} [\Phi(Y_1(\sigma, x)) - \Phi(Y_2(\sigma, x))] d\sigma.$$

This implies that for any $p > 1/\beta$

$$\begin{aligned} |I_3(t \wedge \tau_R)|_E^p &\leq c_{\beta,p} \left(\int_0^{t \wedge \tau_R} (t \wedge \tau_R - s)^{(\beta-1)\frac{p}{p-1}} ds \right)^{p-1} \int_0^{t \wedge \tau_R} I_{\{s < \tau_R\}} |v_\beta(s)|_E^p ds \\ &\leq c_{\beta,p}(t) \int_0^t I_{\{s < \tau_R\}} |v_\beta(s)|_E^p ds. \end{aligned}$$

Now, in view of (4.22), if we assume $\beta < \epsilon$ and $p > 1/\beta \vee 1/(\epsilon - \beta)$, we have

$$\begin{aligned} &\int_0^t I_{\{s < \tau_R\}} |v_\beta(s)|_E^p ds \\ &\leq \int_0^t I_{\{s < \tau_R\}} \left| \int_0^s (s - \sigma)^{-\beta} (\lambda - A) e^{(s-\sigma)A} [\Phi(Y_1(\sigma, x)) - \Phi(Y_2(\sigma, x))] d\sigma \right|^p ds \\ &\leq c_p(\lambda) \int_0^t I_{\{s < \tau_R\}} \left(\int_0^s (s - \sigma)^{-\beta-1+\epsilon} |\Phi(Y_1(\sigma, x)) - \Phi(Y_2(\sigma, x))|_{E_\epsilon} d\sigma \right)^p ds \\ &\leq \int_0^t \left(\int_0^s (s - \sigma)^{-\beta-1+\epsilon} |Y_1(\sigma \wedge \tau_R, x) - Y_2(\sigma \wedge \tau_R, x)|_E d\sigma \right)^p ds \\ &\leq \left(\int_0^t s^{-(1+\beta-\epsilon)\frac{p}{p-1}} ds \right)^{p-1} \int_0^t |Y_1(s \wedge \tau_R, x) - Y_2(s \wedge \tau_R, x)|_E^p ds, \end{aligned}$$

so that

$$|I_3(t \wedge \tau_R)|_E^p \leq c_{\beta,p,\lambda}(t) \int_0^t |Y_1(s \wedge \tau_R, x) - Y_2(s \wedge \tau_R, x)|_E^p ds, \quad (5.21)$$

for some function $c_{\beta,p,\lambda}(t) \downarrow 0$, as $t \downarrow 0$.

Step 4. By a stochastic factorization argument, for any $R > 0$ and $\beta \in (0, 1)$ we have

$$I_4(t \wedge \tau_R) = c_\beta \int_0^{t \wedge \tau_R} (t \wedge \tau_R - s)^{\beta-1} e^{(t \wedge \tau_R - s)A} v_\beta(s) ds,$$

where

$$v_\beta(s) = \int_0^s (s - \sigma)^{-\beta} e^{(s-\sigma)A} [D\Phi(Y_1(\sigma, x)) - D\Phi(Y_2(\sigma, x))] \cdot dw(\sigma).$$

Therefore, if $\epsilon < 2\beta$ and $p > 2/(2\beta - \epsilon) \vee 1/\epsilon$ we have

$$\begin{aligned}
|I_4(t \wedge \tau_R)|_E^p &\leq |I_4(t \wedge \tau_R)|_{W^{\epsilon,p}(0,1)}^p \\
&\leq c_{\beta,p} \left(\int_0^{t \wedge \tau_R} (t \wedge \tau_R - s)^{\beta-1-\frac{\epsilon}{2}} I_{\{s < \tau_R\}} |v_\beta(s)|_{L^p(0,1)} ds \right)^{\frac{1}{p}} \\
&\leq c_{\beta,p} \left(\int_0^{t \wedge \tau_R} (t \wedge \tau_R - s)^{(\beta-1-\frac{\epsilon}{2})\frac{p}{p-1}} ds \right)^{p-1} \int_0^t I_{\{s < \tau_R\}} |v_\beta(s)|_{L^p(0,1)}^p ds \\
&\leq c_{\beta,p}(t) \int_0^t I_{\{s < \tau_R\}} |v_\beta(s)|_{L^p(0,1)}^p ds.
\end{aligned}$$

For any $\xi \in [0, 1]$, we have

$$\begin{aligned}
&I_{\{s < \tau_R\}} |v_\beta(s, \xi)|^p \\
&\leq \left| \sum_{i=1}^{\infty} \int_0^s (s - \sigma)^{-\beta} \left(e^{(s-\sigma)A} [D\Phi(Y_1(\sigma \wedge \tau_R, x)) - D\Phi(Y_2(\sigma \wedge \tau_R, x))] e_i \right) (\xi) d\beta_i(\sigma) \right|^p,
\end{aligned}$$

then, from the Burkholder-Davies-Gundy inequality we have

$$\begin{aligned}
\mathbb{E} I_{\{s < \tau_R\}} |v_\beta(s, \xi)|^p &\leq c_p \mathbb{E} \left(\int_0^s (s - \sigma)^{-2\beta} \right. \\
&\quad \left. \sum_{i=1}^{\infty} \left| \left(e^{(s-\sigma)A} [D\Phi(Y_1(\sigma \wedge \tau_R, x)) - D\Phi(Y_2(\sigma \wedge \tau_R, x))] e_i \right) (\xi) \right|^2 d\sigma \right)^{\frac{p}{2}}.
\end{aligned}$$

Now, thanks to (4.21) we have

$$\begin{aligned}
&\sum_{i=1}^{\infty} \left| \left(e^{(s-\sigma)A} [D\Phi(Y_1(\sigma \wedge \tau_R, x)) - D\Phi(Y_2(\sigma \wedge \tau_R, x))] e_i \right) (\xi) \right|^2 \\
&= \sum_{i=1}^{\infty} |\langle [D\Phi(Y_1(\sigma \wedge \tau_R, x)) - D\Phi(Y_2(\sigma \wedge \tau_R, x))] e_i, K_{s-\sigma}(\xi, \cdot) \rangle_H|^2 \\
&= \|[D\Phi(Y_1(\sigma \wedge \tau_R, x)) - D\Phi(Y_2(\sigma \wedge \tau_R, x))]^* K_{s-\sigma}(\xi, \cdot)\|_H^2 \\
&\leq c (s - \sigma)^{-\frac{1}{2}} |Y_1(\sigma \wedge \tau_R, x) - Y_2(\sigma \wedge \tau_R, x)|_E^2,
\end{aligned}$$

and then

$$\mathbb{E} I_{\{s < \tau_R\}} |v_\beta(s)|_{L^p(0,1)}^p \leq c_p \mathbb{E} \left(\int_0^s (s - \sigma)^{-(2\beta+\frac{1}{2})} |Y_1(\sigma \wedge \tau_R, x) - Y_2(\sigma \wedge \tau_R, x)|_E^2 d\sigma \right)^{\frac{p}{2}}.$$

Hence, if $\beta < 1/4$ and $p > 2/(1 - 4\beta)$, this implies

$$\mathbb{E} \int_0^t I_{\{s < \tau_R\}} |v_\beta(s)|_{L^p(0,1)}^p ds \leq c_{\beta,p}(t) \int_0^t \mathbb{E} |Y_1(s \wedge \tau_R, x) - Y_2(s \wedge \tau_R, x)|_E^p ds,$$

so that

$$\mathbb{E} |I_4(t \wedge \tau_R)|_E^p \leq c_{\beta,p}(t) \int_0^t \mathbb{E} |Y_1(s \wedge \tau_R, x) - Y_2(s \wedge \tau_R, x)|_E^p ds. \quad (5.22)$$

Step 5. Conclusion. From (5.19), (5.20), (5.21) and (5.22), for any $R > 0$ and for any p and λ large enough, we have

$$\begin{aligned} \mathbb{E} |Y_1(t \wedge \tau_R, x) - Y_2(t \wedge \tau_R, x)|_E^p &\leq \frac{1}{2} \mathbb{E} |Y_1(t \wedge \tau_R, x) - Y_2(t \wedge \tau_R, x)|_E^p \\ &+ c_{p,R}(T) \int_0^t \mathbb{E} |Y_1(s \wedge \tau_R, x) - Y_2(s \wedge \tau_R, x)|_E^p ds, \quad t \in [0, T]. \end{aligned}$$

This implies that for any fixed $R > 0$

$$\mathbb{E} |Y_1(t \wedge \tau_R, x) - Y_2(t \wedge \tau_R, x)|_E^p = 0.$$

Therefore, if we take the limit as $R > 0$, since $\tau_R \uparrow T$, \mathbb{P} -a.s. as $R \uparrow +\infty$, we conclude that

$$\mathbb{E} |Y_1(t, x) - Y_2(t, x)|_E^p = 0, \quad t \in [0, T].$$

□

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